ABSTRACT

LOCALIZATION PROPERTIES FOR THE UNITARY ANDERSON MODEL

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APPLIED MATHEMATICS

We study a unitary version of the one-dimensional Anderson model, given by a five diagonal deterministic unitary operator multiplicatively perturbed by a random phase matrix. This operator models the time evolution of an electron in a one-dimensional metal ring subject to a magnetic field that linearly increases with time.

We fully characterize positivity and vanishing of the Lyapunov exponent for this model for arbitrary distributions of the random phases. This includes Bernoulli distributions, where in certain cases a finite number of critical spectral values, with vanishing Lyapunov exponent, exists. Thus, we prove that for all non-trivial distributions the model has no absolutely continuous spectrum.

For non-singular distributions of the random phases, we show strong spectral localization, i.e. the spectrum is pure point almost surely with exponentially decaying eigenfunctions.

Moreover, if the random phases have an absolutely continuous distribution with bounded density, the model is shown to be dynamically localized, i.e. the probability of finding an electron in a high energy state is exponentially small for all time.
DEDICATION

To my beloved family
Mom, Dad, Hanan, Hamada
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Chapter 1

Introduction

In his celebrated paper “Absence of Diffusion in Certain Random Lattices” [5], P. W. Anderson discovered one of the most striking quantum interference phenomena: wave function localization due to presence of disorder. Anderson’s goal was to give a simplified theoretical model to describe transport in random lattices. One system his paper describes is the motion of a single electron in a regular lattice with different nuclear charges randomly distributed among the lattice sites. In dimension one, the discrete Anderson model is given by a family of self adjoint operators $h_\omega$ on $l^2(\mathbb{Z})$ defined as

$$h_\omega = h_0 + \lambda Q_\omega.$$  

$h_0$ is the discrete Laplacian which acts on $u \in l^2(\mathbb{Z})$ as $(h_0 u)(n) = u(n+1) + u(n-1)$, while $Q_\omega$ is a diagonal operator of independent identically distributed (i.i.d.) real valued random variables, $\text{diag}\{q_\omega(n)\}$ and $\lambda$ is a tunable strength parameter.

Anderson [5] argued that if the disorder is sufficiently strong, i.e. large enough $\lambda$, it gives rise to localized states, compared to the extended states in the case of periodic potential. Later, Mott and Twose [33] concluded that in one dimension the presence of any amount of disorder leads to localized states. These pioneering physical insights inspired a great amount of work, some of which aimed at showing that localization effects caused by disorder are not exclusive to this model but that Anderson localization is a phenomenon with a high degree of universality. Indeed, it has been shown that Anderson localization exists in a number of other systems such as electromagnetic waves in a disordered dielectric medium [34] and vibrations in irregular structures [39].

Below, we present another example of such systems, namely a model describing a single electron in a small one-dimensional metal ring in the presence of a large uniform electric field generated by a magnetic field that linearly increases with time [6, 29].
The time evolution of this system is given by a random unitary operator with a matrix representation exhibiting a band structure. This operator on \( l^2(\mathbb{Z}) \) can be written (up to a unitary equivalence) in the form

\[
U_\omega = D_\omega S(t),
\]

where \( S(t) \) is a unitary operator with a five-diagonal matrix representation, which is invariant under translations by two. \( S(t) \) depends on a parameter \( t \in (0, 1) \), which controls the size of its off-diagonal elements, see Chapter 2 below. \( D_\omega \) is a diagonal matrix of random phases, \( \text{diag}\{e^{-i\theta_k}\} \). For our application, \( \{\theta_k^\omega : k \in \mathbb{Z}\} \) is a sequence of i.i.d. random variables with values on the one-dimensional torus \( T = \mathbb{R}/2\pi\mathbb{Z} \) having a non-trivial probability distribution \( \mu \), i.e. \( \text{supp} \mu \) has two or more elements. \( U_\omega \) can be considered as a “unitary Anderson-type model”, where \( S \) plays the role of the free Laplacian and where the perturbation is introduced via multiplication rather than addition to ensure that the resulting operator is still unitary. As \( t \to 0 \), \( S(t) \) tends to the identity operator, whereas as \( t \to 1 \), \( S(t) \) tends to a direct sum of shift operators. Accordingly, the spectrum of \( U_\omega \) is pure point if \( t = 0 \) and purely absolutely continuous if \( t = 1 \) \([9]\). Therefore, the parameter \( t \) takes the role of a disorder parameter for \( U_\omega \).

This model was suggested \([6]\) to study the single particle properties of an electron in a metal ring subject to a magnetic field that increases linearly in time. The adiabatic eigenstates of the system form a complete basis and the field induces transition between these levels (Zener tunnling), thus promoting the particle to higher energy states. The phase randomization generated by this process is the source of disorder in this system.

The model was studied numerically in \([6]\) and the authors concluded that the process of phase randomization alone leads to exponential localization of the electron in energy space. Motivated by these results, the spectral analysis of a class of random unitary operators with similar band structure was first undertaken in \([9]\) and later in \([26], [27], [23] \) and \([24]\). In section 1.2, we briefly recall the physical motivation for this specific construction. For more on the physical background of the model we refer the interested reader to \([6]\) and references therein.
1.1. The Main Results

Due to the specific band structure of the operator $U_\omega$, the generalized eigenvectors can be studied using complex $2 \times 2$ transfer matrices. This formalism allows to introduce the Lyapunov exponent $\gamma(z)$, where $z$ is the spectral parameter, see equation (2.9) below. Due to the fact that the transfer matrices have determinants of unit modulus, the Lyapunov exponent is almost surely non-negative.

Our results can be roughly divided into three parts; first we prove that for all non-trivial distributions of the random phases the model possesses no absolutely continuous spectrum. This is done by investigating the positivity of the Lyapunov exponent $\gamma(z)$, defined by (2.9), and showing that it vanishes at most at two “critical” values of $z$. This implies, using the unitary version of Ishii-Pastur Theorem (Theorem 2.3), that the almost sure absolutely continuous spectrum is empty, irrespective of the underlying probability measure.

In Chapter 3, we fully determine the set $\{ |z| = 1 : \gamma(z) > 0 \}$ for arbitrary distribution $\mu$. For most choices of $\mu$ we find that $\gamma(z) > 0$ for all $|z| = 1$, see Theorem 3.1. Nevertheless, there is one exceptional situation: If $\mu$ is a Bernoulli measure supported on two diametrically opposed points of the torus, i.e. supp $\mu = \{a, b\}$, $|a - b| = \pi$, then there exist two critical quasi-energies $z = e^{-ia}$ and $z = e^{-ib}$ at which the Lyapunov exponent vanishes, while it is positive for all other values of $z$ (Theorem 3.2). Our main tool in proving positivity of the Lyapunov exponent is Fürstenberg’s Theorem [7].

However, in order to prove the existence of the critical points a more detailed study of the behavior of the transfer matrices $T_z(\omega, n)$ (2.7) is required. In the proof of Theorem 3.2, we show that at the anomalies $z = e^{-ia}$ and $z = e^{-ib}$, $T_z(\omega, n)$ satisfies the asymptotics

\begin{equation}
\frac{1}{n} \mathbb{E} \left( (\ln \|T_z(\omega, n)\|)^2 \right) \longrightarrow C > 0,
\end{equation}

i.e., roughly, $\|T_z(\omega, z)\| \sim e^{(Cn)^{1/2}}$. In particular, this means that generalized eigenfunctions are not bounded as in the case of periodic systems.

It is interesting to note that the structure of Lyapunov exponents for the unitary Anderson model is richer than for the self-adjoint one-dimensional Anderson model (1.1).
For the latter it has been long known that the Lyapunov exponent is positive at all energies for all non-trivial single site distributions of the random potential, e.g. [12].

Positivity of $\gamma(z)$ is often considered a strong indication of localization of the system [30]. In the second part of this work we concentrate on the case where the phases have a non-singular distribution, guaranteeing that the Lyapunov exponent is always positive. Under such assumption we are able to prove strong spectral localization for the model (1.2). In particular, we prove that if the distribution $\mu$ of the i.i.d. phases possesses a non-trivial absolutely continuous component, then the spectrum of $U_\omega$ is pure point almost surely, with exponentially decaying eigenfunctions. This is proven, in Chapter 4, using positivity of the Lyapunov exponent along with a unitary version of spectral averaging [14, 8].

Spectral localization of $U_\omega$ has been perviously shown in [27], for arbitrary dimension, under the assumptions that the common distribution of the phases is absolutely continuous and the parameters of the $d$-dimensional deterministic unitary operator $S$ are such that $S$ is close to the identity. This is the unitary analog of the familiar large disorder regime under which localization holds for the $d$-dimensional Anderson model. When applied to the one-dimensional case, this yields localization only if the parameter $t$ is sufficiently small. In the present work we prove localization for all values of the parameter $t \in (0, 1)$, thereby completing the analogy with the self-adjoint one dimensional Anderson model, where localization holds without any additional disorder assumption. This part of the manuscript is an updated version of the results published in [23].

Finally for random phases having purely absolutely continuous distribution with bounded density, we actually show strong dynamical localization, i.e. the probability of finding an electron in a high energy state is exponentially small for all time;

$$
\mathbb{E}[\sup_{n \in \mathbb{Z}} |\langle e_k, [U_\omega]^n e_0 \rangle|] \leq \tilde{C}e^{-\beta|k|},
$$

for all $k \in \mathbb{Z}$. This proves the conjecture [6], that phase randomization alone leads to exponential localization of the electron in energy space independent of the Zener tunneling amplitude $t$. 
Our method of proving dynamical localization requires a detailed analysis of the matrix elements of the resolvent \((U_\omega - z)^{-1}\), i.e. Green’s function, for the set \(\{z \in \mathbb{C} : 1/2 < |z| < 1\}\). We start this analysis in Chapter 5, by developing an explicit formula for those matrix elements in terms of solutions of \((U_\omega - z)\psi = 0\). In the following two chapters, we proceed to prove exponential decay of the expectation of a fractional moment of those elements, more precisely we show that there exists \(s \in (0, 1)\) and \(0 < C, \alpha < \infty\) such that

\[
\mathbb{E}[\langle |e_k,(U_\omega - z)^{-1}e_l| \rangle^s] \leq Ce^{-\alpha|k-l|},
\]

for all \(z\) such that \(0 < |z| - 1| < 1/2\), and \(k, l \in \mathbb{Z}\).

A similar estimate was proven for the multi-dimension unitary Anderson model, in [27] in the large disorder regime, small \(t\), where it was used in proving almost sure pure point spectrum of \(U_\omega\). In Chapter 8 we show that exponential decay of a fractional moment of Green’s function actually leads to strong dynamical localization of the model independent of the dimension. Similar techniques of proving dynamical localization via fractional moment estimates have been perviously used in the self-adjoint case in [5] and [3].

In a more general class of self-adjoint Anderson-type models (dimer models, polymer models), it has been shown that the existence of critical energies with vanishing Lyapunov exponents can lead to the co-existence of spectral localization and suitable forms of dynamical delocalization, e.g. [25]. The simplest self-adjoint model which shows this phenomenon is the so-called dimer model [18, 17], in which the random couplings appear in the form of identical neighboring pairs. The typical anomalies encountered in the dimer model are stronger than those in Theorem 3.2 in the sense that transfer matrices, rather than satisfying (1.3), are uniformly bounded in \(n\) and thus the generalized eigenfunctions are bounded.

In Appendix B we study a unitary version of the dimer model, where we show that the Lyapunov exponent is positive away from at most finitely many critical values. However, for the unitary dimer model with Bernoulli distributed phases, i.e. \(\text{supp } \mu = \{a, b\}\), and such that \(|a - b|\) is in the spectrum of \(S\), there are two critical values where transfer matrices are of the type studied in [25], in particular they are bounded in \(n\). The results
shown in this appendix, as well as those of Chapter 3, are a somewhat expanded version of [24].

1.2. The Physical Motivation

The model we study was proposed by Blatter and Browne [6] in order to study the single particle properties of an electron in a ring threaded by a linear time dependent magnetic flux. This system is described by a time dependent Hamiltonian. Neglecting the curvature of the ring, the instantaneous Hamiltonian corresponds to that of a one dimensional Schrödinger operator with periodic potential and time dependent Floquet type boundary conditions. Both eigenstates and energies are periodic in time and the instantaneous spectrum is given by bands separated by gaps determined by the static scattering potential in the ring. As the electric field increases, it induces transition between the adiabatic energy levels of the system (Zener tunnling). The motion of the electron through energy space can be thought of a series of scattering processes, whenever two energy levels come close, the electron either moves up to the higher energy level or backscatters to the same energy level. The time evolution operator is assumed to couple adjacent pairs of states only by means of this mechanism. Thus, a state indexed by $n$ is coupled once to the state $n - 1$ and once with $n + 1$, as their corresponding eigenvalues become close to one another. This yields the time evolution operator over a period to have a five diagonal band structure when written in the basis of eigenvectors at time zero.

The main question we address for this system is the effect of phase randomization induced by Zener tunnling between energy levels on the single particle behavior of the electron. In order to isolate this effect the system is assumed to have a constant Zener tunnling amplitude $t$.

1.3. Relation to OPUC

Let us finally mention that unitary operators on $l^2(\mathbb{N})$ with the same band structure as $S$ and $U_\omega$ above, also arise in the form of so-called CMV-matrices in the study of orthogonal polynomials on the unit circle, e.g. [35, 10]. These operators are characterized by a
sequence of complex numbers \( \{\alpha_k\}_{k \in \mathbb{N}} \), such that \( |\alpha_k| < 1, \ k \in \mathbb{N} \), called the Verblunsky coefficients. As mentioned in [26, 23], when \( |\alpha_k(\omega)| = r \) for all \( k \in \mathbb{N} \), and only the phases \( \eta_k \) of \( \alpha_k(\omega) \) are random, the CMV matrix is unitarily equivalent to the product \( D_\omega S \) on \( l^2(\mathbb{N}) \) which, modulo boundary conditions at site 0, is of the form considered on \( l^2(\mathbb{Z}) \) above. The phases of Verblunsky coefficients are given in terms of the phases \( \theta_k \) by

\[
\eta_k = \theta_k + \theta_{k-1} + \cdots + \theta_0, \quad k = 0, 1, 2, \ldots
\]

see section 9.1 below. The point now is that the random phases of the coefficients \( \alpha_k(\omega) \) are correlated if the phases of the diagonal matrix \( D_\omega \) are independent.

Therefore we get as a corollary of our general analysis that localization takes place for random CMV-matrices with certain types of correlated Verblunsky coefficients. Let us mention here that previous localization results for CMV-matrices provided in [21], [41], [36], [37], and [40] essentially consider independent Verblunsky coefficients.
CHAPTER 2

THE MODEL

Analogous to the self-adjoint case, we look at a random unitary operator as a random perturbation of a deterministic ("free") unitary operator. The model and the results presented in this section can be found in [9], [23], [26] and [27]. Motivated by [6], we construct the free unitary operator $S$ on $l^2(\mathbb{Z})$ as follows: Let $B_1$ and $B_2$ be unitary $2 \times 2$ matrices given by

$$B_1 = \begin{pmatrix} r & t \\ -t & r \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} r & -t \\ t & r \end{pmatrix},$$

with the real parameters $t$ and $r$ linked by $r^2 + t^2 = 1$ to ensure unitarity. Now let $U_e$ be the unitary operator in $l^2(\mathbb{Z})$ found as the direct sum of identical $B_1$-blocks with blocks starting at even indices. Similarly, construct $U_o$ with identical $B_2$-blocks, where blocks start at odd indices. The operator $S$ is now defined as $S = U_e U_o$. The resulting operator $S$ is unitary, with band structure and is 2-periodic

$$S = \begin{pmatrix} \ddots & rt & -t^2 \\ r^2 & -rt & \iddots \\ rt & r^2 & rt & -t^2 \\ -t^2 & -tr & r^2 & -rt \\ rt & r^2 \\ -t^2 & -tr & \ddots \end{pmatrix},$$

where the position of the origin in $\mathbb{Z}$ is fixed by $\langle e_{2k-2}, S e_{2k} \rangle = -t^2$, with $e_k$ ($k \in \mathbb{Z}$) denoting the canonical basis vectors in $l^2(\mathbb{Z})$. Due to unitary equivalence it suffices to consider $0 \leq t, r \leq 1$. Thus $S$ is determined by $t$. We shall sometimes write $S(t)$ to emphasize this dependence. The spectrum of $S(t)$ is given by the arc

$$\sigma(S(t)) = \Sigma(t) = \{ e^{i\vartheta} : \vartheta \in [-\arccos(1 - 2t^2), \arccos(1 - 2t^2)] \}. $$

8
which is symmetric about the real axis and grows from the single point \( \{1\} \) for \( t = 0 \) to the entire unit circle for \( t = 1 \). Thus, excluding the trivial special cases, we shall assume that \( 0 < t < 1 \) for which the spectrum of \( S(t) \) is purely absolutely continuous, see [9].

The random perturbation is then introduced via multiplication by a diagonal matrix

\[
D_\omega = \text{diag}\{e^{-i\theta_k}\},
\]

with \( \{\theta_k^\omega : k \in \mathbb{Z}\} \) a sequence of i.i.d. random variables on the torus \( \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \). More precisely, we introduce the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \( \Omega \) is identified with \( \mathbb{T}^\mathbb{Z} \), \( \mathcal{F} \) is the \( \sigma \)-algebra generated by cylinders of Borel sets, and \( \mathbb{P} = \bigotimes_{k \in \mathbb{Z}} \mu \), where \( \mu \) is a non trivial probability measure on \( \mathbb{T} \).

The random variables \( \theta_k \) on \((\Omega, \mathcal{F}, \mathbb{P})\) are defined by

\[
\theta_k : \Omega \to \mathbb{T}, \quad \theta_k^\omega = \omega_k, \quad k \in \mathbb{Z}.
\]

This ensures that the resulting operator

\[
U_\omega = D_\omega S
\]

is unitary and ergodic with respect to the 2-shift in \( \Omega \), see section 2.1 below. \( U_\omega \) also inherits the band structure of the original operator \( S \).

**Remark 2.0.1.** (i) In this definition, \( S \) plays the role of the free Laplacian in the self-adjoint Anderson model. Thus \( U_\omega \) may be considered as a “unitary Anderson-type model”.

(ii) Note that the band structure (2.2) is inherited by \( U_\omega \). There is no interesting tridiagonal unitary analog of \( S \) (and thus \( U_\omega \)). In fact, it has been shown [10] that any unitary operator with a matrix representation less than five diagonal is either unitary equivalent to a shift operator or is the infinite direct sum of finite unitary matrices. In both cases the spectral theory of such an operator would be trivial.
(iii) As $t \to 0$, $S(t)$ tends to the identity operator, whereas as $t \to 1$, $S(t)$ tends to a direct sum of shift operators. Accordingly, the spectrum of $U_\omega$ is pure point if $t = 0$ and purely absolutely continuous if $t = 1$ [9].

(iv) Even with the given band structure, our choice of $S$ as the unitary analog of the Laplacian is mainly due to the underlying physical model [6]. One could generalize $S$ by choosing $B_1$ and $B_2$ to be two arbitrary unitary $2 \times 2$-matrices and then proceeding as before to get $S' = U e U_\omega$. Any such $S'$ is unitary, has the same block structure as $S$, and is $2$-periodic.

In order to analyze the spectrum of $U_\omega$, we look at solutions of the eigenvalue equation

$$U_\omega \psi = z \psi, \quad \psi = \sum_{k \in \mathbb{Z}} c_k e_k,$$

with $c_k \in \mathbb{C}$, $\{e_k\}_{k \in \mathbb{Z}}$ is the canonical basis in $l^2(\mathbb{Z})$, and $z \in \mathbb{C} \backslash \{0\}$. These solutions are characterized by the relations

$$
\begin{pmatrix}
  c_{2k+1} \\
  c_{2k+2}
\end{pmatrix}
= T_z(\theta^\omega_{2k}, \theta^\omega_{2k+1})
\begin{pmatrix}
  c_{2k-1} \\
  c_{2k}
\end{pmatrix},

$$

for all $k \in \mathbb{Z}$, where the transfer matrices $T_z : \mathbb{T}^2 \to GL(2, \mathbb{C})$ are defined by

$$
T_z(\theta, \eta) = \begin{pmatrix}
  -e^{-i\eta} & r \left( \frac{e^{i(\theta-\eta)} - e^{-i\eta}}{z} \right) \\
  \frac{r}{t} \left( 1 - \frac{e^{-i\eta}}{z} \right) - z e^{i\theta} t^2 + \frac{r^2}{t^2} \left( 1 + e^{i(\theta-\eta)} - \frac{e^{-i\eta}}{z} \right)
\end{pmatrix}.

$$

Note that $\det T_z(\theta^\omega_{2k}, \theta^\omega_{2k+1}) = e^{i(\theta^\omega_{2k} - \theta^\omega_{2k+1})}$ has modulus one and is independent of $z$.

We have for any $n \in \mathbb{N}$

$$
\begin{pmatrix}
  c_{2n-1} \\
  c_{2n}
\end{pmatrix}
= T_z(\theta^\omega_{2(n-1)}, \theta^\omega_{2(n-1)+1}) \cdots T_z(\theta^\omega_0, \theta^\omega_1)
\begin{pmatrix}
  c_{-1} \\
  c_0
\end{pmatrix}
\equiv T_z(\omega, n)
\begin{pmatrix}
  c_{-1} \\
  c_0
\end{pmatrix},

$$

$$
\begin{pmatrix}
  c_{-2n-1} \\
  c_{-2n}
\end{pmatrix}
= T_z(\theta^\omega_{-2n}, \theta^\omega_{-2n+1})^{-1} \cdots T_z(\theta^\omega_{-2}, \theta^\omega_{-1})^{-1}
\begin{pmatrix}
  c_{-1} \\
  c_0
\end{pmatrix}
\equiv T_z(\omega, -n)
\begin{pmatrix}
  c_{-1} \\
  c_0
\end{pmatrix}.

$$

We also set $T_z(\omega, 0) = I$. 
As shown in [9], for any \( z \in \mathbb{C} \setminus \{0\} \), the Lyapunov exponent

\[
\gamma^\pm_\omega(z) = \lim_{n \to \pm \infty} \frac{1}{|n|} \ln \|T_\omega(z,n)\| \tag{2.8}
\]

almost surely exists, has the same value for \( n \to \infty \) and \( n \to -\infty \), and takes the deterministic value

\[
\gamma(z) = \lim_{n \to -\infty} \frac{\mathbb{E}(\ln \|T_\omega(z,n)\|)}{n} \text{ for a.e. } \omega. \tag{2.9}
\]

### 2.1. Properties of the Model

Our definition (2.5) leads to ergodicity of the unitary operator \( U_\omega \). Indeed, introducing the shift operator \( W \) on \( \Omega \) by

\[
(W\omega)_k = \omega_{k+2}, \quad k \in \mathbb{Z}, \tag{2.10}
\]

we get an ergodic set \( \{W^j\}_{j \in \mathbb{Z}} \) of translations. With the unitary operators \( V_j \) defined on the canonical basis of \( l^2(\mathbb{Z}) \) by

\[
V_j e_k = e_{k-2j}, \quad \forall k \in \mathbb{Z}, \tag{2.11}
\]

we observe that for any \( j \in \mathbb{Z} \)

\[
U_{W^j\omega} = V_j U_\omega V_j^*. \tag{2.12}
\]

Therefore, our random operator \( U_\omega \) is an ergodic unitary operator. The general theory of ergodic operators, as for example presented in [12], chapter V, for the self-adjoint case, carries over to the unitary setting. In particular, it follows that the spectrum of \( U_\omega \) is almost surely deterministic, i.e. there is a subset \( \Sigma \) of the unit circle such that \( \sigma(U_\omega) = \Sigma \) for almost every \( \omega \). The same holds for the absolutely continuous, singular continuous and pure point parts of the spectrum: There are \( \Sigma_{ac}, \Sigma_{sc} \) and \( \Sigma_{pp} \) such that almost surely \( \sigma_{ac}(U_\omega) = \Sigma_{ac}, \sigma_{sc}(U_\omega) = \Sigma_{sc} \) and \( \sigma_{pp}(U_\omega) = \Sigma_{pp} \). Moreover, as shown in [26], we can characterize \( \Sigma \) in terms of the support of \( \mu \) and of the spectrum \( \Sigma(t) \) of \( S(t) \).
Theorem 2.1. Under the above hypotheses, the almost sure spectrum of $U_\omega$ consists of the set

$$\Sigma = \exp(i \text{supp } \mu) \Sigma(t) = \{e^{i\alpha} \Sigma(t) | \alpha \in \text{supp } \mu\}. \tag{2.13}$$

Here $\text{supp } \mu$ denotes the support of the probability measure $\mu$, defined as

$$\text{supp } \mu := \{a | \mu(a - \epsilon, a + \epsilon) > 0 \text{ for all } \epsilon > 0\}.$$ 

The link between the behavior at infinity of the generalized eigenvectors and the spectrum of $U_\omega$ is provided by Sh'nol's Theorem. This is a deterministic fact which, as proven in [9], carries over the the unitary operators considered here (here $E_\omega(\cdot)$ is the spectral resolution of $U_\omega$, which we consider to be supported on $\mathbb{T}$):

Theorem 2.2. $\sigma(U_\omega)$ is the closure of the set

$$S_\omega = \{\alpha \in \mathbb{T}; U_\omega \phi = e^{i\alpha} \phi \text{ has a non-trivial polynomially bounded solution}\} \tag{2.14}$$

and $E_\omega(\mathbb{T} \setminus S_\omega) = 0$.

A version of the Ishii-Pastur theorem suited to unitary matrices with a band structure follows as a corollary to these arguments.

Theorem 2.3. Let $U_\omega$ be defined by (2.5), (2.3) and (2.2) and $\gamma(z)$ by (2.8). Then

$$\Sigma_{ac} \subseteq \{e^{ix} : x \in \overline{\{\lambda : \gamma(e^{i\lambda}) = 0\}}^{\text{ess}}\} \tag{2.15}$$

Here the essential closure of a set $A \subset \mathbb{R}$ is given by $A^{\text{ess}} = \{y \in \mathbb{R} : \text{for all } \epsilon > 0, |A \cap (y - \epsilon, y + \epsilon)| > 0\}$.

In certain cases we will have to deal with a finite (or semi-finite) version of $U_\omega$;

2.2. Finite and Semi-finite Operator

For $\eta \in \mathbb{T}$, the unitary semi-finite volume operator $S^{(2n, \infty)}_\eta$ is constructed as follows:
Let the $2 \times 2$ matrices $B_1$ and $B_2$ be defined as 2.1, then let $U^{(2n, \infty)}_\epsilon$ be the unitary operator in $l^2([2n, \infty))$ found as the direct sum of identical $B_1$-blocks with blocks starting at $2n$. 

Similarly, construct $U_0^{(2n,\infty)}$ starting with a single $1 \times 1$ block such that $(U_0^{(2n,\infty)})(2n, 2n) = e^{i\eta}$, then identical $B_2$-blocks starting at $2n + 1$. Now let $S_\eta^{(2n,\infty)} = U_0^{(2n,\infty)}U_0^{(2n,\infty)}$. The operator $S_\eta^{(2n,\infty)}$ on $l^2([2n, \infty))$ will have a band structure

\[
S_\eta^{(2n,\infty)} = \begin{pmatrix}
re^{i\eta} & rt & -t^2 \\
-tre^{i\eta} & r^2 & -rt \\
rt & r^2 & rt & -t^2 \\
-t^2 & -tr & r^2 & -rt \\
rt & r^2 \\
-t^2 & -tr & & & & & & & & & & \vdots
\end{pmatrix}.
\]

(2.16)

Note that the parameter $\eta$ can be thought of as a sort of boundary condition at $2n$.

Similarly, $U_0^{(-\infty,2n+1]}$ is found as the direct sum of identical $B_1$-blocks with blocks starting at even indices, while $U_0^{(-\infty,2n+1]}$ with identical $B_2$-blocks starting at odd indices, with $(U_0^{(-\infty,2n+1]})(2n + 1, 2n + 1) = e^{i\eta}$. Thus,

\[
S_\eta^{(-\infty,2n+1]} = \begin{pmatrix}
\cdots & rt & -t^2 \\
 & r^2 & -rt \\
rt & r^2 & rt & -t^2 \\
-t^2 & -rt & r^2 & -rt \\
rt & r^2 & te^{i\eta} \\
-t^2 & -rt & & & & & & & & & & \vdots
\end{pmatrix}.
\]

(2.17)

Using the same procedure we obtain

\[
S_\eta^{(-\infty,2n]} = \begin{pmatrix}
\cdots & rt & -t^2 \\
 & r^2 & -rt \\
rt & r^2 & rt & -t^2 \\
-t^2 & -rt & r^2 & -rt \\
 & & & & & & & & & & & \vdots
\end{pmatrix},
\]

(2.18)
while,

\[
S^{(2n+1,\infty)}_\eta = \begin{pmatrix}
re^{i\eta} & -te^{i\eta} \\
rt & r^2 & rt & -t^2 \\
-t^2 & -rt & r^2 & -rt \\
rt & r^2 & -t^2 & -rt \\
& & & \ddots
\end{pmatrix}.
\]

(2.19)

In similar fashion, for any \(-\infty \leq a < b \leq \infty\) we can construct the unitary operator \(S^{[a,b]}_{\eta_a,\eta_b}\) with \(\eta_a\) boundary condition at \(a\) and \(\eta_b\) boundary condition at \(b\), for example, it is easy to see that for \(m \geq n + 2\), we have

\[
S^{[2n,2m]}_{\eta_a,\eta_b} = \begin{pmatrix}
re^{i\eta_n} & rt & -t^2 \\
-te^{i\eta_n} & r^2 & -rt \\
rt & r^2 & rt & -t^2 \\
-t^2 & -rt & r^2 & -rt \\
& & & \ddots \\
& & & & te^{i\eta_m} & re^{i\eta_m}
\end{pmatrix}.
\]

(2.20)

Finally, we define

\[
U^{[a,b]}_{\omega,\eta_a,\eta_b} = D^{[a,b]}_{\omega}S^{[a,b]}_{\eta_a,\eta_b},
\]

(2.21)

where \(D^{[a,b]}_{\omega} = \text{diag}\{e^{-i\theta_j}\} \) with \(\{\theta_j: j \in [a, b]\}\) a sequence of i.i.d. random variables on the torus \(T = \mathbb{R}/2\pi\mathbb{Z}\).

In order to investigate the spectral properties of the semi-finite operators \(U^{[a,b]}_{\omega,\eta_a,\eta_b}\), where either \(a\) or \(b\) is infinite, we introduce the generalized eigenfunctions defined by

\[
U^{[a,b]}_{\omega,\eta_a,\eta_b}\psi = z\psi \quad \psi = \sum_{m \in [a,b]} c_m e_m,
\]

with \(z \in \mathbb{C}\setminus\{0\}\). \(c_m \in \mathbb{C}\) are characterized by the relations

\[
\begin{pmatrix}
c_{2m+1} \\
c_{2m+2}
\end{pmatrix} = T_z(\theta_{2m}^\omega, \theta_{2m+1}^\omega) \begin{pmatrix}
c_{2m-1} \\
c_{2m}
\end{pmatrix},
\]

(2.22)
where the transfer matrix $T_z(\theta, \eta)$ is defined by (2.7) and for $m \in I \subset [a, b]$. The exact definition of the set $I$ depends on whether the finite endpoint is even or odd, in particular we have

$$I = \begin{cases} 
\left[ \frac{a+2}{2}, \infty \right) & a = 2n, b = \infty \\
\left[ \frac{a+1}{2}, \infty \right) & a = 2n + 1, b = \infty \\
(-\infty, \frac{b-3}{2}] & a = -\infty, b = 2n + 1 \\
(-\infty, \frac{b-2}{2}] & a = -\infty, b = 2n.
\end{cases}$$

These relations must be supplemented by appropriate relations connecting the boundary terms;

Case I: For $[2n, \infty)$, the boundary terms are given by

$$\begin{pmatrix} c_{2n+1} \\ c_{2n+2} \end{pmatrix} = c_{2n} \begin{pmatrix} \frac{1}{t} \left( r e^{i(\theta \omega_{2n+1} - \theta \omega_{2n})} - e^{i(\eta \omega_{2n+1} - \theta \omega_{2n})} \right) \\ \frac{1}{t^2} \left( r^2 e^{i(\theta \omega_{2n+1} - \theta \omega_{2n})} - \frac{r e^{i(\eta \omega_{2n+1} - \theta \omega_{2n})} z}{z} - e^{i(\theta \omega_{2n+1} + \eta)} \right) \end{pmatrix}.$$  \hspace{1cm} (2.23)

Case II: For $[2n + 1, \infty)$, we have

$$c_{2n+2} = \frac{1}{t} (r - z e^{i(\theta \omega_{2n+1} - \eta)}) c_{2n+1}.$$  \hspace{1cm} (2.24)

Case III: For $(-\infty, 2n]$, it follows that

$$c_{2n-1} = -\frac{1}{t} (r - z e^{i(\theta \omega_{2n} - \eta)}) c_{2n}.$$  \hspace{1cm} (2.25)

Case IV: In the case $(-\infty, 2n + 1]$, the boundary terms are connected through

$$\begin{pmatrix} c_{2n-1} \\ c_{2n} \end{pmatrix} = c_{2n+1} \begin{pmatrix} \frac{1}{t^2} \left( r^2 e^{i(\theta \omega_{2n+1} + \theta \omega_{2n})} - \frac{r e^{i(\eta \omega_{2n+1})} z}{z} - z e^{i\theta \omega_{2n+1}} + r e^{i\eta} \right) \\ -\frac{1}{t} \left( r e^{i(\theta \omega_{2n+1} - \theta \omega_{2n})} - \frac{e^{i(\eta \omega_{2n+1} - \theta \omega_{2n})}}{z} \right) \end{pmatrix}. $$  \hspace{1cm} (2.26)

In each of those cases the corresponding forward (or backward) Lyapunov exponent $\gamma^{[a, b]}(z)$ corresponding to (2.22), defined by (2.8), exists almost surely \cite{20} and the conclusions of Theorems 2.2 and 2.3 remain true for the semi-finite operator $U_{i[a,b]}^{[\omega, \eta_a, \eta_b]}$ \cite{9}.
3.1. Positivity versus Vanishing

We begin this chapter by giving a complete characterization of the sets of spectral parameters where the Lyapunov exponent is positive (zero) for the different types of probability distributions. In [9] the case where the random phases are uniformly distributed on $\mathbb{T}$ where studied and positivity of the Lyapunov exponent is assessed by means of Fürstenberg’s Theorem for all values inside the spectrum. In the present case, the support of $\mu$ is arbitrary and more detailed investigation is necessary.

All norms on $GL(n, \mathbb{C})$ being equivalent, we choose to work with the row-sum norm for convenience. Thus, in what follows the norm is the maximum row sum, i.e. for $A = (a_{ij})_{i,j=1}^{n}$, $||A|| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$. $P(\mathbb{C}^2)$ denotes the projective space of $\mathbb{C}^2$, we write $\bar{v} \in P(\mathbb{C}^2)$ for the direction of $v \in \mathbb{C}^2 \setminus \{0\}$. The action of a $2 \times 2$ matrix $A$ on $P(\mathbb{C}^2)$ is defined by $A\bar{v} = \overline{Av}$.

Our results about the positivity of the Lyapunov exponent can be stated as follows

**Theorem 3.1.** If $\{a, b\} \subset \text{supp } \mu$ such that $|a - b| \notin \{0, \pi\}$, then for every $z \in \mathbb{C} \setminus \{0\}$ we have $\gamma(z) > 0$. In particular, if $\text{supp } \mu$ contains at least three elements, then $\gamma(z) > 0$ for all $z \in \mathbb{C} \setminus \{0\}$.

On the other hand, for a particular choice of the underlying distribution of the random phases the unitary Anderson model, unlike the self-adjoint one, exhibits two critical values of the spectral parameter, where the Lyapunov exponent vanishes.

**Theorem 3.2.** If $\text{supp } \mu = \{a, b\}$ and $|a - b| = \pi$, then

(i) $\gamma(e^{-ia}) = \gamma(e^{-ib}) = 0$;

(ii) $\gamma(z) > 0$, for all $z \in \mathbb{C} \setminus \{e^{-ia}, e^{-ib}, 0\}$. 

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The fact that for all non-trivial probability measures \( \mu \), the set of critical quasi-energies contains at most two points combined with Theorem 2.3 gives the following immediate corollary, concerning the almost sure absolutely continuous spectrum of \( U_\omega \);

**Corollary 3.1.** For any non-trivial distribution \( \mu \) of the i.i.d. random phases, we have \( \Sigma_{ac} = \emptyset \).

We give the proofs of Theorem 3.1 and 3.2 in the following two sections. The last section is dedicated to discussing the continuity of the Lyapunov exponent, where we prove that, away from the critical points, \( \gamma(z) \) is a continuous function of the spectral parameter.

### 3.2. Critical Quasi-energies

In this section we prove Theorem 3.2(i), i.e. that a Bernoulli measure \( \mu \) with diametrically opposed masses \( a, b \) indeed gives rise to two critical quasi-energies at \( z = e^{-ia}, z = e^{-ib} \). Denote \( \mu(a) = p \in (0,1) \) and \( \mu(b) = q = 1 - p \).

For \( z = e^{-ia} \), the i.i.d. random matrices \( T_z(\theta_{2k}^\omega, \theta_{2k+1}^\omega) \) take only the following values with non-zero probabilities,

\[
T_{e^{-ia}}(\theta_{2k}^\omega, \theta_{2k+1}^\omega) =
\begin{cases}
T_{e^{-ia}}(a,a) = -I, & \text{with probability } p^2 \\
T_{e^{-ia}}(b,b) = \begin{pmatrix}
1 & 2r/t \\
2r/t & (3r^2 + 1)/t^2
\end{pmatrix}, & \text{with probability } q^2 \\
T_{e^{-ia}}(b,a) = \begin{pmatrix}
-1 & -2r/t \\
0 & 1
\end{pmatrix}, & \text{with probability } pq \\
T_{e^{-ia}}(a,b) = \begin{pmatrix}
1 & 0 \\
2r/t & -1
\end{pmatrix}, & \text{with probability } pq.
\end{cases}
\]
The latter matrices take much simpler forms when represented with respect to the basis \( \{ \begin{pmatrix} 1 \\ (r+1)/t \end{pmatrix}, \begin{pmatrix} 1 \\ (r-1)/t \end{pmatrix} \} \) of \( \mathbb{C}^2 \). Hence, we define the matrices \( A(\theta, \eta) \) as

\[
A_z(\theta, \eta) := \begin{pmatrix} 1 & 1 \\ (r+1)/t & (r-1)/t \end{pmatrix}^{-1} T_z(\theta, \eta) \begin{pmatrix} 1 & 1 \\ (r+1)/t & (r-1)/t \end{pmatrix},
\]

It follows that

\[
A_{e^{-ia}}(a, a) = -I, \quad A_{e^{-ia}}(b, b) = \begin{pmatrix} \rho & 0 \\ 0 & 1/\rho \end{pmatrix},
\]

\[
A_{e^{-ia}}(b, a) = \begin{pmatrix} 0 & -1/\rho \\ -\rho & 0 \end{pmatrix}, \quad A_{e^{-ia}}(a, b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

with \( \rho := (r+1)^2/t^2 > 1 \).

A straightforward calculation shows that

\[
\gamma(z) = \lim_{n \to \infty} \frac{E(\ln ||\Lambda_z(\omega, n)||)}{n},
\]

where \( \Lambda_z(\omega, n) = \prod_{k=1}^{n} A_z(\theta_{2k}, \theta_{2k+1}) \). In order to simplify the notation, we will suppress the \( e^{-ia} \) dependence of various quantities for the remainder of the section.

Let \( u_0 := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( u_n(\omega) = \begin{pmatrix} u_{n,1}(\omega) \\ u_{n,2}(\omega) \end{pmatrix} := \Lambda(n, \omega) u_0 \).

**Lemma 3.1.** If \( x_n^\omega = \frac{\ln |u_{n,1}(\omega)|}{\ln \rho} \) for \( n \geq 0 \), then

\[
\gamma(e^{-ia}) = \ln \rho \lim_{n \to \infty} \frac{1}{n} E(||x_n(\omega)||).
\]

**Proof.** From (3.2) it follows that with probability one there are \( x^\omega, y^\omega \in \mathbb{R} \) with \( x^\omega y^\omega = 1 \) and either \( \Lambda(n, \omega) = \begin{pmatrix} x^\omega & 0 \\ 0 & y^\omega \end{pmatrix} \) or \( \Lambda(n, \omega) = \begin{pmatrix} 0 & x^\omega \\ y^\omega & 0 \end{pmatrix} \). In both cases it follows readily that

\[
||\Lambda(n, \omega)|| = ||\Lambda(n, \omega) u_0||_{\infty},
\]

where \( || \cdot ||_{\infty} \) denotes the max-norm on \( \mathbb{C}^2 \). This implies that

\[
\gamma(e^{-ia}) = \lim_{n \to \infty} \frac{1}{n} E(\ln ||\Lambda(n, \omega) u_0||_{\infty}).
\]
Furthermore, we see from the specific form of \( \Lambda(n) \) that \( \Lambda(n, \omega) u_0 = \begin{pmatrix} u_{n,1}(\omega) \\ 1/u_{n,1}(\omega) \end{pmatrix} \). Therefore

\[
\ln ||\Lambda(n, \omega) u_0||_\infty = \ln \max(|u_{n,1}(\omega)|, \frac{1}{|u_{n,1}(\omega)|})
\]

\[
= \ln |u_{n,1}(\omega)|.
\]

The required result then follows from (3.5) and the definition of \( x_n^\omega \). \( \square \)

The following lemma is devoted to the necessary analysis of the random sequence \( x_n^\omega \).

**Lemma 3.2.** \( (x_n^\omega)_{n \geq 0} \) is an integer-valued Markov chain with \( x_0^\omega = 0 \) and transition probabilities

\[
\mathbb{P}(x_{n+1}^\omega = x_n^\omega) = p^2, \quad \mathbb{P}(x_{n+1}^\omega = x_n^\omega + 1) = q^2,
\]

\[
\mathbb{P}(x_{n+1}^\omega = -x_n^\omega) = pq, \quad \mathbb{P}(x_{n+1}^\omega = -(x_n^\omega + 1)) = pq.
\]

**Proof.** Clearly, \( x_0^\omega = 0 \) for all \( \omega \). Let \( A(n+1, \omega) := A_{c=1}((\theta_2(n+1), \theta_2(n+1)) \). In the case \( A(n+1, \omega) = -I \), we have \( |u_{n+1,1}(\omega)| = |u_{n,1}(\omega)| \), i.e. \( x_{n+1}^\omega = x_n^\omega \). If \( A(n+1, \omega) = \begin{pmatrix} \rho & 0 \\ 0 & 1/\rho \end{pmatrix} \), then \( |u_{n+1,1}(\omega)| = \rho |u_{n,1}(\omega)| \) and \( x_{n+1}^\omega = x_n^\omega + 1 \). Similarly, \( A(n+1, \omega) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) implies that \( |u_{n+1,1}(\omega)| = |u_{n,2}(\omega)| = 1/|u_{n,1}(\omega)| \) and thus \( x_{n+1}^\omega = -x_n^\omega \). Finally, \( A(n+1, \omega) = \begin{pmatrix} 0 & -1/\rho \\ -\rho & 0 \end{pmatrix} \) gives

\[
|u_{n+1,1}(\omega)| = \frac{1}{\rho} |u_{n,2}(\omega)| = \frac{1}{\rho |u_{n,1}(\omega)|},
\]

i.e. \( x_{n+1}^\omega = -x_n^\omega - 1 \). Thus \( x_{n+1}^\omega \) is determined by \( x_n^\omega \) and \( A(n+1, \omega) \). The transition probabilities follow from (3.1) and (3.2). \( \square \)

**Lemma 3.3.** As \( n \to \infty \),

\[
\frac{\mathbb{E}((x_n^\omega)^2)}{n} \to \frac{q}{2p}.
\]
In particular, we have that for all $n$,

$$\mathbb{E}(|x_n^\omega|) \leq Cn^{1/2}.$$ 

**Proof.** Let $\alpha = q - p$, we denote by $\mathbb{E}(x|y)$ the conditional expectation of $x$ given $y$. It follows that

$$\mathbb{E}(x_n^\omega|x_{n-1}^\omega) = \alpha^2 x_{n-1} + q\alpha.$$ 

Since $\mathbb{E}(x_n^\omega) = \mathbb{E}(\mathbb{E}(x_n^\omega|x_{n-1}^\omega))$, we get that

$$\mathbb{E}(x_n^\omega) = \alpha^2 \mathbb{E}(x_{n-1}) + q\alpha$$

and, iterating that

$$\mathbb{E}(x_n^\omega) = \alpha^{2n} \mathbb{E}(x_0^\omega) + q\alpha \frac{1 - \alpha^{2n}}{1 - \alpha^2}.$$ 

Similarly, since

$$\mathbb{E}((x_n^\omega)^2|x_{n-1}^\omega) = (x_{n-1}^\omega)^2 + 2qx_{n-1}^\omega + q,$$

we have that

$$\mathbb{E}((x_n^\omega)^2) = \mathbb{E}((x_{n-1}^\omega)^2) + 2q\mathbb{E}(x_{n-1}^\omega) + q.$$ 

Another induction gives that, for all $n \geq 3$,

$$\mathbb{E}((x_n^\omega)^2) = \mathbb{E}((x_0^\omega)^2) + 2q \frac{1 - \alpha^{2n}}{1 - \alpha^2} \mathbb{E}(x_0^\omega) + nq(1 + \frac{2q\alpha}{1 - \alpha^2})$$

$$+ 2q^2\alpha[1 - \frac{1}{1 - \alpha^2}(2 + \alpha^4 \frac{1 - \alpha^{2(n-2)}}{1 - \alpha^2})].$$ 

Since $|\alpha| < 1$, $\mathbb{E}(x_0^\omega) = \mathbb{E}((x_0^\omega)^2) = 0$, and $q(1 + \frac{2q\alpha}{1 - \alpha^2}) = q/2p$, we get (3.7). Using that

$$\mathbb{E}(|x_n^\omega|) \leq \mathbb{E}((x_n^\omega)^2)^{1/2},$$

in turn, proves the second assertion and finishes the proof. \(\Box\)

Lemma 3.3 is, in fact, a consequence of general extensions of the Central Limit Theorem used in the study of dynamical systems, e.g. Section A.4 of [13]. We include the previous elementary proof for the convenience of the reader.

The main result of this section now follows immediately.
Proof of Theorem 3.2(i). The fact that $\gamma(e^{-ia}) = 0$ follows directly from (3.4) and Lemma 3.3. The proof of $\gamma(e^{-ib}) = 0$ is identical. □

3.3. Positivity of the Lyapunov Exponent

In this section we show that, except for the two critical energies discussed above, the Lyapunov exponent (2.9) is positive. This constitutes the contents of part (ii) of Theorem 3.2 and of Theorem 3.1. These results are proven using F"urstenberg’s Theorem [20].

For each $z \in \mathbb{C}$, the random variables $\theta_0^z$ and $\theta_1^z$ induce a measure on $GL(2, \mathbb{C})$ through $T_z(\theta_0^z, \theta_1^z)$. Denote the smallest closed subgroup of $GL(2, \mathbb{C})$ generated by the support of this measure by $G_{z,\mu}$. Thus $G_{z,\mu}$ is generated by the matrices $T_z(\theta, \eta)$, defined in (2.7), where $\theta$ and $\eta$ vary in supp $\mu$.

F"urstenberg’s Theorem [20] states that if $G_{z,\mu}$ is non-compact and strongly irreducible, then

$$\gamma(z) = \lim_{n \to \infty} \frac{\mathbb{E}(|\ln ||T_z(\omega, n)|||)}{n} > 0.$$

First we prove that $G_{z,\mu}$ is non-compact for all values of $z \in \mathbb{C}\setminus\{0\}$.

**Lemma 3.4.** $G_{z,\mu}$ is non-compact for all $z \in \mathbb{C}\setminus\{0\}$.

**Proof.** Let $\theta$ and $\eta$ be in supp $\mu$, $\theta \neq \eta$, and let $x := e^{-i\theta}/z$, $y := e^{-i\eta}/z$. Define

(3.8) $D := T_z(\theta, \theta)T_z(\theta, \eta)^{-1} = \begin{pmatrix} x/y & 0 \\ \frac{r}{t}(x/y - 1) & 1 \end{pmatrix} \in G_{z,\mu},$

(3.9) $E := T_z(\eta, \theta)^{-1}T_z(\theta, \theta) = \begin{pmatrix} 1 & \frac{r}{t}(1 - y/x) \\ 0 & \frac{r}{t}(1 - y/x) \end{pmatrix} \in G_{z,\mu},$

(3.10) $L := DE = \begin{pmatrix} x/y & \frac{r}{t}(x/y - 1) \\ \frac{r}{t}(x/y - 1) & y/x - \frac{r^2}{t^2}|x/y - 1|^2 \end{pmatrix} \in G_{z,\mu},$

(3.11) $J := ED = \begin{pmatrix} x/y - \frac{r^2}{t^2}|y/x - 1|^2 & \frac{r}{t}(1 - y/x) \\ \frac{r}{t}(1 - y/x) & y/x \end{pmatrix} \in G_{z,\mu}.$
Note that \( \det L = \det J = 1 \) and that \( J^{-1} = L^* \). Thus we get the self-adjoint element \( K := J^{-1}L \) of \( G_{z,\mu} \). In fact, \( K \) is positive definite and \( \det K = 1 \). More calculation shows that
\[
\text{tr} K = 1 + \frac{2r^2}{t^2} |x/y - 1|^2 + \left| x/y - \frac{r^2}{t^2} |x/y - 1|^2 \right|^2
= 2 + \frac{r^2}{t^4} |x/y - 1|^4.
\]

As \( \theta \neq \eta \) and therefore \( x/y \neq 1 \) we conclude that \( \text{tr} K > 2 \). Positivity of \( K \) implies that it has an eigenvalue strictly bigger than 1. Thus, containing all powers of \( K \), the group \( G_{z,\mu} \) is non-compact for all \( z \in \mathbb{C}\setminus\{0\} \).

It remains to prove strong irreducibility under the assumptions of Theorem 3.2(ii) as well as under those of Theorem 3.1. Under the already established non-compactness of \( G_{z,\mu} \), strong irreducibility of \( G_{\lambda,\mu} \) is equivalent to
\[
(3.12) \quad \# \{ g\bar{v} : g \in G_{z,\mu} \} \geq 3 \quad \text{for all } \bar{v} \in \mathbf{P}(\mathbb{C}^2);
\]
see [7]. First we deal with the case where the support of \( \mu \) contains two points of \( T \) that are not diametrically opposed.

**Proof of Theorem 3.1.** Each element of the projective space \( \mathbf{P}(\mathbb{C}^2) \) is of the form \( \bar{v} \) with \( v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) or \( v = \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \), for some \( \alpha \in \mathbb{C} \). In terms of \( x, y \), introduced above, the condition that \( |a - b| \notin \{0, \pi\} \) can be written as \( x/y \notin \{-1, 1\} \) while \( |x/y| = 1 \).

**Case I:** Let \( v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). The action of the operator \( E \), defined in (3.9), on \( \bar{v} \) has the direction of \( \begin{pmatrix} r(t^{-1}x - 1) \\ 1 \end{pmatrix} \), while \( E^2\bar{v} \) has the direction of \( \begin{pmatrix} r(t^{-1}(x/y)^2 - 1) \\ 1 \end{pmatrix} \). Thus \( \{I, E, E^2\} \subset G_{z,\mu} \) maps \( \bar{v} \) into three different elements in \( \mathbf{P}(\mathbb{C}^2) \).

**Case II:** Let \( v = \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \), with \( \alpha \in \mathbb{C} \). Acting on \( \bar{v} \) with the operator \( D \) from (3.8) results in the direction of \( \begin{pmatrix} 1 \\ r(t^{-1}(1 - y/x) + \alpha y/x) \end{pmatrix} \).
Defining the map $F$ such that $F : c \mapsto \frac{r}{t}(1 - y/x) + cy/x$, one sees that $F$ has a single fixed point at $c = r/t$. Since the second iteration $F^2 : c \mapsto \frac{r}{t}(1 - (y/x)^2) + c(y/x)^2$, has the same value $r/t$ as its only fixed point, we deduce that $\{c, F(c), F^2(c)\}$ are pairwise different except when $c = r/t$. Thus iterations of the operator $D$ take $\bar{v}$ into at least three different directions, unless $\alpha = r/t$.

On the other hand, $E\bar{v}$ has the direction of
\[
\begin{pmatrix}
1 \\
\frac{1}{\alpha y/x} \\
\frac{1}{1 + \alpha r/t(1 - y/x)}
\end{pmatrix},
\]
and the map $H : c \mapsto \frac{c y/x}{1 + c r/t(1 - y/x)}$ has fixed points $0, -t/r$, which are also the fixed points of $H^2 : c \mapsto \frac{c(y/x)^2}{1 + c r/t(1 - (y/x)^2)}$. In particular, $\{I, E, E^2\} \subset G_{z,\mu}$ map the direction vector \[
\begin{pmatrix}
1 \\
\frac{1}{r/t}
\end{pmatrix}
\]
to three different elements in $P(\mathbb{C}^2)$. This proves the required condition (3.12) for strong irreducibility of $G_{z,\mu}$ and the result of Lemma 3.4 and Fürstenberg’s Theorem finish the proof.

Next we prove that for $\text{supp} \, \mu = \{a, b\}$ and $|a - b| = \pi$, $e^{-ia}$ and $e^{-ib}$ are the only critical quasi-energies.

**Proof of Theorem 3.2(ii).** Let $\theta = a, \eta = b$. In the terminology introduced above, the condition that $|a - b| = \pi$ and $z \notin \{e^{-ia}, e^{-ib}, 0\}$ can be written as $-x = y \notin \{-1, 1\}$. Since $x = -y$, the operator $L \in G_{z,\mu}$ defined by (3.10) takes the form

\[
L = \begin{pmatrix}
-1 & -2r \\
-2r & \frac{t}{t} \\
\frac{t}{t} & -1 - \frac{4r^2}{t^2}
\end{pmatrix}.
\]

As $\det L = 1$ and $|\text{tr}L| > 2$, $L$ is hyperbolic, hence iterations of $L$ map any direction in $P(\mathbb{C}^2)$ to infinitely many directions, except when $\bar{v}$ coincides with the direction of one of its eigenvectors, given by $v_+ = \left(\frac{r + 1}{t} \right), v_- = \left(\frac{1}{r - 1} \right)$.

Next we prove that even for the eigenvectors of $L$, we have that $\#\{g\bar{v}_+ : g \in G_{z,\mu}\} \geq 3$. 

Under the current conditions, the transfer matrices take the form

\[ T_z(\theta, \theta) = \begin{pmatrix} -x & \frac{r}{t}(1-x) \\ \frac{r}{t}(1-x) & \frac{r^2}{t^2} (1-2x) \end{pmatrix}, \]

\[ T_z(\theta, \eta) = \begin{pmatrix} x & \frac{r}{t}(-1+x) \\ \frac{r}{t}(1+x) & \frac{r^2}{t^2} x - \frac{1}{t^2 x} \end{pmatrix}, \]

\[ T_z(\eta, \theta) = \begin{pmatrix} -x & -\frac{r}{t} (1+x) \\ \frac{r}{t}(1-x) & -\frac{r^2}{t^2} x + \frac{1}{t^2 x} \end{pmatrix}, \]

\[ T_z(\eta, \eta) = \begin{pmatrix} x & \frac{r}{t} (1+x) \\ \frac{r}{t}(1+x) & \frac{r^2}{t^2} (2+x) + \frac{1}{t^2 x} \end{pmatrix}. \]

Therefore, we have that

\[ T_z(\theta, \theta)v_+ = \begin{pmatrix} -r + \frac{1}{t^2} (r-x) \\ -\frac{r+1}{t^3} (rx+1/x) + \frac{r(r+1)^2}{t^3} \end{pmatrix}, \]

\[ T_z(\theta, \eta)v_+ = \begin{pmatrix} -r + \frac{1}{t^2} (r-x) \\ \frac{r+1}{t^3} (rx-1/x) + \frac{r}{t} \end{pmatrix}, \]

\[ T_z(\eta, \theta)v_+ = \begin{pmatrix} -r + \frac{1}{t^2} (r+x) \\ -\frac{r+1}{t^3} (rx-1/x) + \frac{r}{t} \end{pmatrix}, \]

\[ T_z(\eta, \eta)v_+ = \begin{pmatrix} -r + \frac{1}{t^2} (r+x) \\ \frac{r+1}{t^3} (rx+1/x) + \frac{r(r+1)^2}{t^3} \end{pmatrix}. \]

A simple calculation shows that \( T_z(\theta, \theta)v_+ = T_z(\theta, \eta)v_+ \) only if \( x = r \). Similarly, \( T_z(\theta, \eta)v = T_z(\eta, \theta)v \) only if \( x \in \{-1, 1\} \), while assuming that \( T_z(\theta, \theta)v_+ = T_z(\eta, \theta)v_+ \) is equivalent to \( (r+1)t^2 = 0 \). Therefore, under the assumptions of Theorem 3.2(ii), for \( x \neq r \) we have that \( \#\{T_z(\theta, \theta)v_+, T_z(\theta, \eta)v_+, T_z(\eta, \theta)v_+\} = 3 \). In the case \( x = r \), \( T_z(\theta, \theta)v_+ \)}.
has the direction of the vector \( \begin{pmatrix} 0 \\ -t/r \end{pmatrix} \), which in turn can be mapped to infinitely many directions in \( \mathbf{P}(\mathbb{C}^2) \) using iterations of \( L \). We thus have proven that

\[
\#\{g\bar{v} : g \in G_{z,\mu}\} \geq 3 \quad \text{for all } \bar{v} \in \mathbf{P}(\mathbb{C}^2).
\]

Combining this with Lemma 3.4, Fürstenberg’s Theorem gives the required assertion. □

### 3.4. Continuity of the Lyapunov Exponent

In this section we prove that, away from the critical points, the Lyapunov exponent is a continuous function of the spectral parameter \( z \). The proof of this fact is similar to the one given in [12] for the self-adjoint case.

First, for a compact set \( \Lambda \) of quasi-energies (not including zero) with positive Lyapunov exponents, we define the function

\[
\Phi(z, \bar{v}) = \mathbb{E}(\ln \frac{||T_z v||}{||v||}), \quad \bar{v} \in \mathbf{P}(\mathbb{C}^2), \ z \in \Lambda.
\]

Where \( T_z := T_z(\theta_{2n}, \theta_{2n+1}) \) denotes the transfer matrix defined in (2.7) with the dependance on \( \omega \) being suppressed in order to simplify the notation. The next lemma establishes a couple of properties of \( \Phi(z, \bar{v}) \).

**Lemma 3.5.** (i) The mapping \( \bar{v} \mapsto \Phi(z, \bar{v}) \) is continuous on \( \mathbf{P}(\mathbb{C}^2) \).

(ii) There exists a constant \( C \) such that,

\[
\sup_{\bar{v} \in \mathbf{P}(\mathbb{C}^2)} |\Phi(z, \bar{v}) - \Phi(z_1, \bar{v})| \leq C|z - z_1|, \quad z, z_1 \in \Lambda.
\]

**Proof.** (i) From (2.7) one sees that the norm of \( T_z \) is uniformly bounded for all \( \omega \) and \( z \in \Lambda \). Consequently, we also have a uniform bound on \( \Phi(z, \bar{v}) \). The assertion is then obtained using the dominated convergence theorem.

(ii) First we note that for all \( \bar{v} \in \mathbf{P}(\mathbb{C}^2) \),

\[
|\Phi(z, \bar{v}) - \Phi(z_1, \bar{v})| \leq \mathbb{E}(\ln \frac{||T_z v||}{||T_{z_1} v||}).
\]
Since $|\det T_z T_{z_1}^{-1}| = 1$, it follows that

$$|\Phi(z, \bar{v}) - \Phi(z_1, \bar{v})| \leq \mathbb{E}(\ln ||T_z T_{z_1}^{-1}||).$$

(3.13)

On the other hand, one has

$$||T_z T_{z_1}^{-1}|| \leq ||T_{z_1}^{-1}|| ||T_z - T_{z_1}|| + 1.$$

Since all norms on $GL(2, \mathbb{C})$ are equivalent, there exists a constant $C_1$ such that

$$||T_z T_{z_1}^{-1}|| \leq C_1 ||T_z - T_{z_1}||_F + 1,$$

(3.14)

where $||A||_F^2 \equiv \sum_{i,j=1}^{2} |a_{ij}|^2$ denotes the Frobenius norm of the matrix $A = \{a_{ij}\}_{i,j=1}^{2}$. From (2.7), it is easy to see that $||T_z - T_{z_1}||_F \leq C_2 |z - z_1|$. Combining this with (3.14) and (3.13) gives the required result. \hfill \Box

Before proving the main result of this section, we recall a general fact: If $\mu_z$ denotes the probability measure on $GL(2, \mathbb{C})$ induced by $T_z$ and $G_{z,\mu}$ is non-compact and strongly irreducible for all $z \in \Lambda$, then there exists a unique distribution $\nu_z$ on $\mathbb{P}(\mathbb{C}^2)$ that is invariant with respect to $\mu_z$. A proof of this fact can be found in [7]. Moreover, we have

**Lemma 3.6.** For $z \in \Lambda$, the mapping $z \mapsto \nu_z$ is weakly continuous.

**Proof.** We start by showing that if the sequence $\{z_n\}_{n \in \mathbb{Z}}$ converges to $z$, then the corresponding measures $\mu_{z_n}$ converge weakly to $\mu_z$. First recall that for all $\omega \in \Omega$, $||T_{z_n}(\omega) - T_z(\omega)|| \leq d_n$, where $d_n = C |z_n - z|$ for some $C \in \mathbb{R}$. Now, let $B \subset GL(2, \mathbb{C})$ such that the boundary of $B$ has zero measure with respect to $\mu_z$, i.e. $\mu_z(\partial B) = 0$. For such a set we have that

$$|\mu_{z_n}(B) - \mu_z(B)| = |\mathbb{P}[T_{z_n}^{-1}(B)] - \mathbb{P}[T_z^{-1}(B)]|$$

$$\leq \mathbb{P}[T_{z_n}^{-1}(B) \cap (T_z^{-1}(B))^c] + \mathbb{P}[T_z^{-1}(B) \cap (T_{z_n}^{-1}(B))^c].$$

It is not difficult to see that

$$\mathbb{P}[T_{z_n}^{-1}(B) \cap (T_z^{-1}(B))^c] \leq \mathbb{P}\{\omega \in \Omega : T_z(\omega) \in B^c, \text{dist}(T_z(\omega), \partial B) \leq d_n\}$$

$$\leq \mu_z\{A : \text{dist}(A, \partial B) \leq d_n\}.$$
Taking the limit as \( n \to \infty \) and using dominated convergence one sees that

\[
\lim_{n \to \infty} \mathbb{P}[T_{z_n}^{-1}(B) \cap (T_{z_n}^{-1}(B))^c] = \mu_z(\partial B) = 0.
\]

Using a similar argument one gets that \( \mathbb{P}[T_{z}^{-1}(B) \cap (T_{z_n}^{-1}(B))^c] \to 0 \) as \( n \to \infty \). Therefore, we have

\[
\lim_{n \to \infty} \mu_{z_n}(B) = \mu_z(B).
\]

Since this is true for any set \( B \) with \( \mu_z(\partial B) = 0 \), weak convergence of \( \mu_{z_n} \) to \( \mu_z \) follows [43]. In order to get the weak convergence of \( \nu_{z_n} \), we use the fact that the set of invariant measures on \( \mathcal{P}(\mathbb{C}^2) \) is compact in the weak* topology [43]; thus every subsequence of \( \{\nu_{z_n}\} \) has a weakly convergent subsequence and since the limit of each of those subsequences is invariant with respect to \( \mu_z \) it equals \( \nu_z \) by uniqueness of the latter. A short contradiction argument shows that \( \nu_{z_n} \) has to converge weakly to \( \nu_z \). \( \square \)

The Lyapunov exponent can be expressed in terms of the mapping \( \Phi \) and the measure \( \nu_z \) as

\[
\gamma(z) = \int \Phi(z, \bar{v}) d\nu_z(\bar{v}).
\] (3.15)

Now we are ready to prove that for any compact set \( \Lambda \) for which \( G_{z,\mu} \) is non-compact and strongly irreducible for all \( z \in \Lambda \), we have

**Theorem 3.3.** (i) The Lyapunov exponent \( \gamma(z) \) is a continuous function of \( z \in \Lambda \).

(ii) Moreover,

\[
\gamma(z) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}\left\{ \ln \frac{||T_z(\omega, n)\bar{v}||}{||\bar{v}||} \right\}
\]

uniformly in \( z \in \Lambda \) and \( \bar{v} \in \mathcal{P}(\mathbb{C}^2) \).
Proof. Let \( z \in \Lambda \) and choose \( \{z_n\} \subset \Lambda \) a sequence of quasi-energies such that \( z_n \to z \) as \( n \to \infty \). It follows that
\[
\lim_{n \to \infty} \gamma(z_n) = \lim_{n \to \infty} \int \Phi(z_n, \bar{v})d\nu_{z_n}(\bar{v})
\]
\[
= \lim_{n \to \infty} \left[ \int \Phi(z, \bar{v})d\nu_{z_n}(\bar{v}) + \int (\Phi(z_n, \bar{v}) - \Phi(z, \bar{v}))d\nu_{z_n}(\bar{v}) \right]
\]
\[
= \int \Phi(z, \bar{v})d\nu_z(\bar{v}),
\]
with the last equality following from Lemma 3.6 along with part (ii) of Lemma 3.5.

(ii) Let \( \nu_{n,z,\bar{v}} = \frac{1}{n} \sum_{k=0}^{n-1} \mu_z^k \ast \delta_{\bar{v}} \) be a sequence of probability measures on \( P(\mathbb{C}^2) \). If \( \bar{v}_n \to \bar{v} \) in \( P(\mathbb{C}^2) \) and \( z_n \to z \) in \( \Lambda \) then it is easy to check that any limit point of \( \nu_{n,z,\bar{v}_n} \) is \( \mu_z \) invariant thus equals to \( \nu_z \). Therefore, \( \nu_{n,z,\bar{v}} \) converges weakly to \( \nu_z \).

Next we define
\[
h_n(z, \bar{v}) = \frac{1}{n} \mathbb{E}\left\{ \ln \frac{||T_z(\omega, n)v||}{||v||} \right\},
\]
it is straightforward to see that \( h_n(z, \bar{v}) = \nu_{n,z,\bar{v}}(\Phi(z)) \). Also
\[
\lim_{n \to \infty} h_n(z_n, \bar{v}_n) = \lim_{n \to \infty} \nu_{n,z,\bar{v}_n}(\Phi(z_n))
\]
\[
= \nu_z(\Phi(z)) = \gamma(z).
\]
Thus, we have that \( h_n(z_n, \bar{v}_n) \) converges to \( \gamma(z) \). Using an argument similar to the one used in the proof of Lemma 3.5 gives that \( h_n \) is continuous on \( \Lambda \times P(\mathbb{C}^2) \). Therefore, we have the required uniform convergence on \( \Lambda \times P(\mathbb{C}^2) \).

\[\square\]

Corollary 3.1. Let \( U_{\omega, \eta_k, \eta_l} \) be defined by (2.21) with either \( k \) or \( l \) infinite and let \( \gamma^{[k,l]} \) be the corresponding Lyapunov exponent defined by (2.8), then the results of Theorems 3.1, 3.2 and 3.3 hold for the \( \gamma^{[k,l]} \).

Proof. This follows using the exact same proofs as in the infinite case. \[\square\]
CHAPTER 4

SPECTRAL LOCALIZATION FOR NON-SINGULAR DISTRIBUTIONS

In this chapter, we deal with the situation where the random phases have a non-singular distribution. We prove that in such cases the model is spectrally localized. Spectral localization of the model, in all dimensions, has been shown in [27] for absolutely continuous distribution and small values of \( t \). Below, we show that in one-dimension the distribution of the random phases need only be non-singular in order to have strong spectral localization for all values of \( 0 < t < 1 \). We need that \( \mu \) has an a.c. component due to the use of a spectral averaging argument in the proof.

We further show in Section 4.2 that our main result is also true for unitary matrices with a similar structure defined on \( l^2([a, \infty)) \) or \( l^2((-\infty, b]) \), see Theorem 4.2. In the proof below we will use the concept of cyclic subspace which is defined as:

**Definition 1.** Let \( h \) be a normal operator on Hilbert space \( \mathcal{H} \). A set \( C \subset \mathcal{H} \) is called a cyclic subset if linear combinations of the elements of \( \{h^n x : n \in \mathbb{Z}, x \in C\} \) are dense in \( \mathcal{H} \).

4.1. The Full Lattice Case

The main result of this section can be stated as follows:

**Theorem 4.1.** If the distribution \( \mu \) of the i.i.d. phases possesses a non-trivial absolutely continuous component, then \( U_\omega \) is pure point almost surely, with exponentially decaying eigenfunctions.

In order to prove this Theorem, we adapt an argument which, in various forms, has been used extensively in localization proofs for various types of one-dimensional random Schrödinger operators. It combines positivity of the Lyapunov exponent with polynomial
boundedness of generalized eigenfunctions (Theorem 2.2) and spectral averaging. While implicit in the literature even earlier, this strategy was first explicitly spelled out in [38].

Let \( \overline{\Omega} = T^{\mathbb{Z}\setminus\{-1,0\}} \), \( \overline{P} = \otimes_{k \in \mathbb{Z}\setminus\{-1,0\}} \mu \) and \( \overline{\omega} = (\ldots, \overline{\omega}_{-3}, \overline{\omega}_{-2}, \overline{\omega}_{1}, \overline{\omega}_{2}, \ldots) \). We will write \((\overline{\omega}, \theta_{-1}, \theta_{0})\) for \((\ldots, \overline{\omega}_{-3}, \overline{\omega}_{-2}, \theta_{-1}, \theta_{0}, \overline{\omega}_{1}, \overline{\omega}_{2}, \ldots)\).

By construction \( \gamma_{(\overline{\omega}, \theta_{-1}, \theta_{0})}(e^{i\alpha}) \) is independent of \((\theta_{-1}, \theta_{0})\). It follows from Theorem 3.1 that for any \( \alpha \in \mathbb{T} \), there exists \( \overline{\Omega}(\alpha) \subset \overline{\Omega} \) with \( \overline{P}(\overline{\Omega}(\alpha)) = 1 \) such that

\[
\gamma_{(\overline{\omega}, \theta_{-1}, \theta_{0})}(e^{i\alpha}) = \gamma(e^{i\alpha}) > 0
\]

for all \((\theta_{-1}, \theta_{0})\) and all \( \overline{\omega} \in \overline{\Omega}(\alpha) \). Hence, by Fubini applied to \( \overline{P} \times |\cdot| \), we get the existence of \( \overline{\Omega}_0 \in \overline{\Omega} \) with \( \overline{P}(\overline{\Omega}_0) = 1 \) such that for every \( \overline{\omega} \in \overline{\Omega}_0 \) there is \( A_{\overline{\omega}} \in \mathbb{T} \) with \( |A_{\overline{\omega}}| = 0 \) and

\[
\gamma_{(\overline{\omega}, \theta_{-1}, \theta_{0})}(e^{i\alpha}) > 0 \quad \text{for all } (\theta_{-1}, \theta_{0}) \text{ and all } \alpha \in A_{\overline{\omega}}^C.
\]

Here \(|\cdot|\) denotes Lebesgue-measure on \( \mathbb{T} \).

Let us show that \( A_{\overline{\omega}}^C \) is a support of the spectral resolution of \( U_{(\overline{\omega}, \theta_{-1}, \theta_{0})} \) for Lebesgue almost every \((\theta_{-1}, \theta_{0})\).

We introduce the spectral measures \( \rho_{\omega}^j \) associated with \( U_{\omega} = \int_{\mathbb{T}} e^{i\alpha} dE_{\omega}(\alpha) \) defined for all \( j \in \mathbb{Z} \) and all Borel sets \( \Delta \in \mathbb{T} \) by

\[
\rho_{\omega}^j(\Delta) = \langle e_j, E_{\omega}(\Delta)e_j \rangle.
\]

By construction, the variation of a random phase at one site is described by a rank one perturbation. More precisely, dropping the subscript \( \omega \) temporarily, we define \( \hat{D} \) by taking \( \theta_0 = 0 \) in the definition of \( D \):

\[
\hat{D} = D + (1 - e^{-i\theta_0}) P_0,
\]

so that, with the obvious notations,

\[
\hat{U} = \hat{D} S.
\]
As for rank one perturbations of self-adjoint operators, a spectral averaging formula holds in the unitary case. In particular, see Proposition 8.1 of [8], for any \( f \in L^1(\mathbb{T}) \),

\[
\int_{\mathbb{T}} d\theta_0 \int_{\mathbb{T}} f(\alpha) d\rho^0_{(\overline{\varepsilon}, \theta_1, \theta_0)}(\alpha) = \int_{\mathbb{T}} f(\alpha) \, d\alpha.
\]

By applying this to the characteristic function of \( A_{\overline{\varepsilon}} \) we get

\[
0 = |A_{\overline{\varepsilon}}| = \int_{\mathbb{T}} \rho^0_{(\overline{\varepsilon}, \theta_1, \theta_0)}(A_{\overline{\varepsilon}}) \, d\theta_0,
\]

implying that \( \rho^0_{(\overline{\varepsilon}, \theta_1, \theta_0)}(A_{\overline{\varepsilon}}) = 0 \) for every \( \theta_1 \) and Lebesgue-a.e. \( \theta_0 \). Similarly, we get \( \rho^{-1}_{(\overline{\varepsilon}, \theta_1, \theta_0)}(A_{\overline{\varepsilon}}) = 0 \) for every \( \theta_0 \) and Lebesgue-a.e. \( \theta_1 \).

Therefore, for all \( \overline{\varepsilon} \in \overline{\Omega}_0 \), there exists \( J_{\overline{\varepsilon}} \subset \mathbb{T}^2 \) such that \( |J_{\overline{\varepsilon}}^C| = 0 \) and

\[
(\theta_1, \theta_0) \in J_{\overline{\varepsilon}} \Rightarrow \rho^j_{(\overline{\varepsilon}, \theta_1, \theta_0)}(A_{\overline{\varepsilon}}) = 0, \ j \in \{-1, 0\}.
\]

Fix \( \overline{\varepsilon} \in \overline{\Omega}_0 \) and \( (\theta_1, \theta_0) \in J_{\overline{\varepsilon}} \) and consider \( \omega = (\overline{\varepsilon}, \theta_1, \theta_0) \). Below we prove that

**Lemma 4.1.** The subspace \( \text{Span} \{e_{-1}, e_0\} \) is cyclic for \( U_\omega \).

Therefore, we deduce from (4.8) that \( E_\omega(A_{\overline{\varepsilon}}) = 0 \). If \( S_\omega \) is the set from Sh’nol’s Theorem 2.2, we conclude that \( S_\omega \cap A_{\overline{\varepsilon}}^C \) is a support for \( E_\omega(\cdot) \).

Let \( \alpha \in S_\omega \cap A_{\overline{\varepsilon}}^C \). By Theorem 2.2, \( U_\omega \psi = e^{i\alpha} \psi \) has a non-trivial polynomially bounded solution \( \psi \). On the other hand, by (4.2), \( \gamma_\omega(e^{i\alpha}) > 0 \). Thus, by Osceledec’s Theorem, every solution which is polynomially bounded at \( +\infty \) necessarily has to decay exponentially, and the same holds at \( -\infty \). Thus \( \psi \) decays exponentially at \( +\infty \) and \( -\infty \), and therefore is in \( l^2(\mathbb{Z}) \) and an eigenfunction of \( U_\omega \). We have shown that every \( \alpha \in S_\omega \cap A_{\overline{\varepsilon}}^C \) is an eigenvalue of \( U_\omega \). As \( l^2(\mathbb{Z}) \) is separable, it follows that \( S_\omega \cap A_{\overline{\varepsilon}}^C \) is countable. Therefore \( E_\omega(\cdot) \) has countable support and thus \( U_\omega \) is pure point spectrum. In particular

\[
\sigma_{ac}(U_\omega) = \emptyset \quad \text{for every } \omega \in \Omega_0 := \{ (\overline{\varepsilon}, \theta_1, \theta_0) : \overline{\varepsilon} \in \overline{\Omega}_0, (\theta_1, \theta_0) \in J_{\overline{\varepsilon}} \}.
\]

From \( |J_{\overline{\varepsilon}}^C| = 0 \) and the non-triviality of the a.c. component of \( \mu \) we have

\[
(\mu \times \mu)(J_{\overline{\varepsilon}}) \geq (\mu_{ac} \times \mu_{ac})(J_{\overline{\varepsilon}}) = (\mu_{ac} \times \mu_{ac})(\mathbb{T}^2) > 0.
\]
As $\mathbb{P}(\Omega_0) = 1$, we conclude from (4.9) and (4.10) that

\begin{equation}
(4.11) \quad \mathbb{P}(\sigma_{sc}(U_\omega) = \emptyset) \geq \mathbb{P}(\Omega_0) = \int_{\Omega_0} d\mathbb{P}(\omega)(\mu \times \mu)(J_\omega) > 0.
\end{equation}

Since the spectral types are almost surely deterministic, thus $\Sigma_{ac} = \emptyset$. We already know $\Sigma_{ac} = \emptyset$ from (3.1). This proves that $U_\omega$ has almost surely pure point spectrum.

We still need to show that almost surely all eigenfunctions decay exponentially. To this end, note that we actually have shown above that the event “all eigenvectors of $U_\omega$ decay at the rate of the Lyapunov exponent” has positive probability (as this is true for all $\omega \in \Omega_0$). For the case of ergodic one-dimensional Schrödinger operators Kotani and Simon show in Theorem A.1 of [28] that this event has probability 1 or 0. In fact, only measurability needs to be shown as the event is invariant under the ergodic transformation $W$. The proof of this fact provided in [28] carries over to our model. Let us only note that, due to Lemma 4.1, we may use $\rho_\omega = \rho_\omega^{-1} + \rho_\omega^0$ as spectral measures in their argument.

This completes the proof of Theorem 4.1 up to the following proof;

Proof of Lemma 4.1: We drop the sub(super)scripts $\omega$ in this proof. We have to show that any vector $e_k, k \in \mathbb{Z}$ can be written as a linear combination of the vectors $U^n e_j, n \in \mathbb{Z}, j = -1$ and $j = 0$. We compute from (2.5) and its adjoint

\begin{align}
(4.12) \quad Ue_{-1} &= e^{-i\theta - 2}re_{-2} + e^{-i\theta - 1}r^2e_{-1} + e^{-i\theta}re_0 - e^{-i\theta_1}t^2e_1, \\
(4.13) \quad Ue_0 &= -e^{-i\theta - 2}t^2e_{-2} - e^{-i\theta - 1}re_{-1} + e^{-i\theta_0}r^2e_0 - e^{-i\theta_1}rte_1, \\
(4.14) \quad U^{-1}e_0 &= e^{i\theta_0}(re_{-1} + r^2e_0 + rte_1 - t^2e_2), \\
(4.15) \quad U^{-1}e_{-1} &= e^{i\theta_1}(-t^2e_{-3} - rte_{-2} + r^2e_{-1} - rte_2).
\end{align}

Hence, using $t^2 + r^2 = 1$,

\begin{align}
(4.16) \quad e_1 &= \frac{e^{i\theta_1}}{t}(e^{-i\theta_0}re_0 - tUe_{-1} - rUe_0), \\
(4.17) \quad e_{-2} &= \frac{e^{i\theta_2}}{t}(rUe_{-1} - tUe_0 - e^{-i\theta_1}re_{-1}).
\end{align}

Therefore, using (4.16) in (4.14), suitable linear combinations of $e_{-1}, e_0, Ue_{-1}, Ue_0$ and $U^{-1}e_0$ yield $e_2$. Similarly, $e_{-3}$ can be obtained as a linear combination of $e_{-1}, e_0, Ue_{-1}, Ue_0$.
and $U^{-1}e_{-1}$ using (4.17) in (4.15). These manipulations lead us from the indices $(-1,0)$ to the set $(1,2)$ in one direction and $(-3,-2)$ in the other direction. Due to the shape of $U$, we can iterate the process to reach any vector.

4.2. The Half-lattice Case

Let $U_{\omega,\eta_a}^{[a,b]}$ be defined as in 2.21, with either $a$ or $b$ being infinite. In this section we indicate how to adapt the results above to the operator $U_{\omega,\eta_a}^{[a,b]}$. In order to simplify the statements of the results, let’s assume without loss of generality that $b = \infty$ since the case when $a = -\infty$ is almost identical. The first difference/simplification with respect to the operator $U_\omega$ defined on the whole lattice is that $U_{\omega,\eta_a}^{[a,\infty]}$ admits a cyclic vector.

**Lemma 4.2.** The vector $e_a$ is cyclic for $U_{\omega,\eta_a}^{[a,\infty]}$.

**Proof.** For $a = 2n$: one has

$$e_{2n+1} = \frac{e^{-i(\eta_{2n} - \theta_{2n+1})}}{t} \{ r e^{i(\eta_{2n} - \theta_{2n})} e_{2n} - U_{\omega,\eta_a}^{[a,\infty]} e_{2n} \}.$$  

While for $a = 2n + 1$, we have

$$e_{2n+2} = \frac{e^{i(\eta_{2n+1} - \theta_{2n+1})}}{t} \{ r e^{-i(\eta_{2n+1} - \theta_{2n+1})} e_{2n+1} - [U_{\omega,\eta_a}^{[a,\infty]}]^{-1} e_{2n+1} \}.$$  

Once those facts are established, the rest of the proof follows as in the proof of Lemma 4.1. □

As a consequence, we get:

**Theorem 4.2.** Let $U_{\omega,\eta_a}^{[a,\infty]}$ be defined by 2.21. If the distribution $\mu$ of the i.i.d. phases possesses a non-trivial absolutely continuous component, then $U_{\omega,\eta_a}^{[a,\infty]}$ is pure point almost surely, with exponentially decaying eigenfunctions.

**Proof.** The proof is virtually the same as that of Theorem 4.1. Due to cyclicity of $e_a$ it suffices to average over the single phase $\theta_a$, which leads to some simplifications. □
CHAPTER 5

GREEN’S FUNCTION

One of the most important objects in spectral theory is the resolvent \((U_\omega - z)^{-1}\) for \(z \in \rho(U_\omega)\). In this chapter we give an explicit formula for Green’s function \(G_z(k,l) = \langle e_k, (U - z)^{-1} e_l \rangle\) in terms of solutions of \((U - z)\psi = 0\). Note that we suppress the \(\omega\) dependence of the various qualities since the randomness plays no role in the following discussion.

5.1. Green’s Function of the Infinite Volume Operator

For a solution \(\varphi\) of \((U - z)\psi = 0\), we define \(\tilde{\varphi}\) by

\[
\begin{pmatrix}
\tilde{\varphi}_{2n} \\
\tilde{\varphi}_{2n+1}
\end{pmatrix} =
\begin{pmatrix}
t^2 & rt \\
rt & r^2 - ze^{i\theta_{2n}}
\end{pmatrix}
\begin{pmatrix}
\varphi_{2n-1} \\
\varphi_{2n}
\end{pmatrix}.
\]

Since \(\varphi\) is a solution of \((U - z)\psi = 0\) this definition is equivalent to

\[
\begin{pmatrix}
\tilde{\varphi}_{2n} \\
\tilde{\varphi}_{2n+1}
\end{pmatrix} =
\begin{pmatrix}
rt^2 - ze^{i\theta_{2n+1}} & -rt \\
-rt & t^2
\end{pmatrix}
\begin{pmatrix}
\varphi_{2n+1} \\
\varphi_{2n+2}
\end{pmatrix}.
\]

A straightforward calculation shows that \(\tilde{\varphi}\) is characterized by the relations

\[
\begin{pmatrix}
\tilde{\varphi}_{2k} \\
\tilde{\varphi}_{2k+1}
\end{pmatrix} = \tilde{T}_z(\theta_{2k-1}, \theta_{2k})
\begin{pmatrix}
\tilde{\varphi}_{2k-2} \\
\tilde{\varphi}_{2k-1}
\end{pmatrix},
\]

for all \(k \in \mathbb{Z}\), where the transfer matrices \(\tilde{T}_z : \mathbb{T}^2 \rightarrow GL(2, \mathbb{C})\) are defined by

\[
\begin{pmatrix}
e^{-i\theta} \\
e^{i(\eta - \theta)} - e^{-i\theta}
\end{pmatrix}
\begin{pmatrix}
t^2(1 - e^{-i\theta}) \\
t^2(e^{i(\eta - \theta)} - e^{-i\theta}) - ze^{i\eta} + r^2(1 + e^{i(\eta - \theta)} - e^{-i\theta})
\end{pmatrix}.
\]

Note that \(\det \tilde{T}_z(\theta_{2k-1}, \theta_{2k}) = e^{i(\theta_{2k} - \theta_{2k-1})}\) is independent of \(z\).
For \( z \in \rho(U) \setminus \{0\} \), let \( G_z(k, l) = \langle e_k, (U - z)^{-1} e_l \rangle \) and let \( \varphi^+, \varphi^- \) be solutions of \((U - z)\psi = 0\) in \( l^2_\pm(\mathbb{Z}) \) respectively, i.e. for all \( r \in \mathbb{Z} \)
\[
\sum_{n=-\infty}^{\infty} |\varphi^+_n|^2 < \infty \text{ and } \sum_{n=-\infty}^{\infty} |\varphi^-_n|^2 < \infty.
\]

**Remark 5.1.1.** (i) Note that \( \varphi^\pm \) are uniquely determined up to a constant. This can be seen using the following simple contradiction argument. If \( \varphi^1, \varphi^2 \) be two linearly independent \( l^2_+ \) solutions, then
\[
\begin{pmatrix}
\varphi^1_{2n-1} & \varphi^2_{2n-1} \\
\varphi^1_{2n} & \varphi^2_{2n}
\end{pmatrix} = T_z(\omega, n) \begin{pmatrix}
\varphi^1_{-1} & \varphi^2_{-1} \\
\varphi^1_0 & \varphi^2_0
\end{pmatrix}.
\]
Taking the determinant of both sides and using that \( |\det T_z(\omega, n)| = 1 \), we get a contradiction.

(ii) A simple calculation shows that \( \tilde{\varphi}^\pm \), defined from \( \varphi^\pm \) using (5.1), are also in \( l^2_\pm(\mathbb{Z}) \).

The following theorem allows us to express Green’s function in terms of \( \varphi^+ \) and the corresponding \( \tilde{\varphi}^+ \), defined in (5.1).

**Theorem 5.1.** If \( l = 2n \) or \( l = 2n + 1 \), then the matrix elements of \( G_z \) are given by
\[
G_z(k, l) = c_l \begin{cases}
\tilde{\varphi}^+_l \varphi^-_k & k < l \text{ or } k = l = 2n \\
\tilde{\varphi}^-_l \varphi^+_k & k > l \text{ or } k = l = 2n + 1,
\end{cases}
\]

where \( c_l = \frac{e^{i\theta_l}}{\tilde{\varphi}^+_{2n+1} \varphi^+_2 - \varphi^-_{2n} \tilde{\varphi}^-_{2n+1}} \).

**Proof.** In what follows we show that the matrix \( G_z \) whose elements are given by (5.5) is indeed the inverse of \( U - z \).

First we verify that \([(U - z)G_z(., l)](k) = \delta_{kl} \). For \( m \in \mathbb{Z} \), let
\[
I_l = \begin{pmatrix}
[(U - z)G_z(., l)](2m) \\
[(U - z)G_z(., l)](2m + 1)
\end{pmatrix} = \begin{pmatrix}
e^{-i\theta_{2m}} [rtG_z(2m-1,l) + (r^2 - ze^{i\theta_{2m}})G_z(2m,l) + rtG_z(2m+1,l) - t^2G_z(2m+2,l)] \\
e^{-i\theta_{2m+1}} [-t^2G_z(2m-1,l) - rtG_z(2m,l) + (r^2 - ze^{-i\theta_{2m+1}})G_z(2m+1,l) - rtG_z(2m+2,l)]
\end{pmatrix}.
\]
Using equation (5.5) we see that for \( l \leq 2m - 1 \)
\[
I_l = \begin{pmatrix}
    c_l \overline{\varphi}_l^{-1} [(U - z) \varphi^+] (2m) \\
    c_l \overline{\varphi}_l^{-1} [(U - z) \varphi^+] (2m + 1)
\end{pmatrix} = 0,
\]
where the last quality follows from the fact that \((U - z) \varphi^+ = 0\). Similarly, for \( l \geq 2m + 2 \),
we get \( I_l = 0 \). That leaves the two cases \( l = 2m, 2m + 1 \).

For \( l = 2m \)
\[
I_{2m} = c_{2m} \begin{pmatrix}
    e^{-i\theta_{2m}} [\varphi_2^+(r^2 - z^2 e^{i\theta_{2m}}) \varphi_2^-(rt \varphi^+_{2m+1} - t^2 \varphi^+_{2m+2})] \\
    e^{-i\theta_{2m+1}} [\varphi_2^-(rt \varphi^+_{2m} - t^2 \varphi^+_{2m+1})]
\end{pmatrix} = c_{2m} \begin{pmatrix}
    e^{-i\theta_{2m}} [\varphi_2^+ - \varphi_2^- \varphi_2^+ \varphi_2^-] \\
    e^{-i\theta_{2m+1}} [-\varphi_2^- \varphi_2^+ + \varphi_2^- \varphi_2^+]
\end{pmatrix} = \begin{pmatrix}
    1 \\
    0
\end{pmatrix},
\]
where the second equality follows from (5.1),(5.2) and the definition of \( c_{2m} \). Similarly,
\[
I_{2m+1} = \begin{pmatrix}
    0 \\
    1
\end{pmatrix}.
\]
Therefore,
\[
[(U - z) G_z (., l)] (k) = \delta_{kl}.
\]
Finally, using the definition of \( G_z (., l) \) and that \( \varphi^\pm \in l^2_\pm (\mathbb{Z}) \), we see that \( G_z e_l \in l^2(\mathbb{Z}) \), for all \( l \in \mathbb{Z} \). Thus, \((U - z) G_z e_l = e_l\), which implies that \( G_z (k, l) = \langle e_k, (U - z)^{-1} e_l \rangle \) for all \( z \in \rho(U) \setminus \{0\} \). \( \Box \)

Remark 5.1.2. A straightforward calculation shows that the expression given in Theorem 5.1 is independent of the choice of the initial conditions of the solutions. More precisely, if \( \varphi^{\pm,1} \) and \( \varphi^{\pm,2} \) are distinct \( l^2_\pm \) solutions and \( G^1_z, G^2_z \) are the respective operators given by (5.5), then \( G^1_z = G^2_z \).

5.2. Green’s Function of the Finite Volume Operator

Let \( U_{\omega, a, b}^\eta \) be defined as in (2.21). A solution \( \phi \) of \((U - z) \psi = 0\) is said to satisfy the \( \eta_a \)-left boundary condition at \( a \), if \((U_{\omega, a, b}^\eta - z) \phi)(j) = 0 \) for all \( j \geq a \). Also, it is said to satisfy the \( \eta_b \)-right boundary condition at \( b \), if \((U_{\omega, a, b}^\eta - z) \phi)(j) = 0 \) for all \( j \leq b \).
Let \( \varphi^a \) be a solution satisfying the \( \eta_a \)-left boundary condition at \( a \), and let \( \varphi^b \) be a solution satisfying the \( \eta_b \)-right boundary condition at \( b \). Define

\[
G_z^{[a,b]} = (U_\omega^{[a,b]} - z)^{-1},
\]

where we suppressed the \( \omega \) dependence in order to simplify the notation. The following theorem gives an expression of the elements of \( G_z^{[a,b]} \) in terms of \( \varphi^a \) and \( \varphi^b \) and the corresponding \( \tilde{\varphi}^a, \tilde{\varphi}^b \) defined as in (5.1).

**Theorem 5.2.** For \( a \leq k, l \leq b \). If \( l = 2n \) or \( l = 2n + 1 \), then

\[
G_z^{[a,b]}(k, l) = c_l \begin{cases} 
\tilde{\varphi}^b \varphi^a & k < l \text{ or } k = l = 2n \\
\tilde{\varphi}^a \varphi^b & k > l \text{ or } k = l = 2n + 1,
\end{cases}
\]

where \( c_l = \frac{e^{i\theta_l}}{\varphi_{2n+1} \varphi_{2n} - \tilde{\varphi}_{2n} \tilde{\varphi}_{2n+1}} \).

**Proof.** Due to the fact that the boundary elements of \( U^{[a,b]}_\omega \) depends on whether \( a, b \) are even or odd, we have to deal with each case separately.

**Case I:** \( a = 2n, b = 2m \) with \( m \geq n + 2 \). Using the definition of \( U^{[a,b]}_\omega \), it is easy to see that

\[
[(U^{[2n,2m]}_\eta_n,\eta_m - z)G_z^{[2n,2m]}(\cdot,l)](k) = [(U - z)G_z^{[2n,2m]}(\cdot,l)](k),
\]

for all \( 2n \leq l \leq 2m \) and \( 2n + 1 < k < 2m \). A computation, similar to that given in the proof of Theorem 5.1, shows that \([(U - z)G_z^{[2n,2m]}(\cdot,l)](k) = \delta_{kl} \) for all \( 2n \leq l \leq 2m \) and \( 2n + 1 < k < 2m \).

In what follows we deal with the boundary cases, i.e. \( k \in \{2n, 2n + 1, 2m\} \).

(a) For \( k = 2n \), we have

\[
[(U^{[2n,2m]}_\eta_n,\eta_m - z)G_z^{[2n,2m]}(\cdot,l)](2n) = c_le^{-i\theta_{2n}} \{ (re^{i\theta_n} - ze^{i\theta_{2n}})G_z^{[2n,2m]}(2n, l) \\
+ rtG_z^{[2n,2m]}(2n + 1, l) - t^2G_z^{[2n,2m]}(2n + 2, l) \}.
\]

Thus, for all \( 2n + 2 \leq l \leq 2m \), it is straightforward to see that

\[
[(U^{[2n,2m]}_\eta_n,\eta_m - z)G_z^{[2n,2m]}(\cdot,l)](2n) = c_l \varphi_{2m}((U^{[2n,2m]}_\eta_n,\eta_m - z)\varphi^{2n})(2n) = 0.
\]
While for \( l = 2n \) or \( l = 2n + 1 \), we have that

\[
[(U_{\eta_n, \eta_m}^{[2n, 2m]} - z)G_z^{[2n, 2m]}(\cdot, l)](2n) = c_l e^{-i\theta_{2n}} \left\{ (re^{i\eta_n} - ze^{i\theta_{2n}})\varphi_l^{2m} \varphi_{2n}^{2n} \right. \\
+ \left. \varphi_l^{2n} (r\varphi_{2n+1} - t^2\varphi_{2n+2}) \right\}.
\]

Since \( \varphi^{2n} \) satisfies the boundary condition at \( 2n \), it follows that

\[
(re^{i\eta_n} - ze^{i\theta_{2n}})\varphi_{2n}^{2n} + r\varphi_{2n+1} - t^2\varphi_{2n+2} = 0.
\]

Using this along with the definition of \( \tilde{\varphi} \) (5.2), we obtain

\[
[(U_{\eta_n, \eta_m}^{[2n, 2m]} - z)G_z^{[2n, 2m]}(\cdot, l)](2n) = c_l e^{-i\theta_{2n}} \left\{ \tilde{\varphi}_l^{2m} \varphi_{2n+1} - \varphi_l^{2n} \varphi_{2n+1} \right\} = \delta_{l,2n}.
\]

Thus, we have shown that

\[
[(U_{\eta_n, \eta_m}^{[2n, 2m]} - z)G_z^{[2n, 2m]}(\cdot, l)](k) = \delta_{kl},
\]

for \( k = 2n \) and all \( 2n \leq l \leq 2m \).

(b) \( k = 2n + 1 \): this case is treated similarly to \( k = 2n \).

(c) In the case \( k = 2m \), we have

\[
[(U_{\eta_n, \eta_m}^{[2n, 2m]} - z)G_z^{[2n, 2m]}(\cdot, l)](2m) = c_l e^{-i\theta_{2m}} \left\{ t e^{i\eta_m} G_z^{[2n, 2m]}(2m - 1, l) \\
+ (re^{i\eta_m} - ze^{i\theta_{2m}})G_z^{[2n, 2m]}(2m, l) \right\}.
\]

Therefore,

\[
[(U_{\eta_n, \eta_m}^{[2n, 2m]} - z)G_z^{[2n, 2m]}(\cdot, l)](2m) = c_l e^{-i\theta_{2m}} \varphi_l^{2m} \left\{ t e^{i\eta_m} \varphi_{2m-1} + (re^{i\eta_m} - ze^{i\theta_{2m}})\varphi_{2m} \right\}
\]

\[
= 0,
\]

for all \( 2n \leq l < 2m \).

Next, we explore the situation when \( l = 2m \). In this case we have

\[
(5.9) \quad [(U_{\eta_n, \eta_m}^{[2n, 2m]} - z)G_z^{[2n, 2m]}(\cdot, 2m)](2m) = c_{2m} e^{-i\theta_{2m}} \varphi_{2m} \left\{ t e^{i\eta_m} \varphi_{2m-1} + (re^{i\eta_m} - ze^{i\theta_{2m}})\varphi_{2m} \right\}.
\]
Using the definition of $\tilde{\varphi}^{2n}$ (5.1), it follows that
\[ t e^{i\eta_m} \varphi^{2n}_{2m-1} + (r e^{i\eta_m} - z e^{i\eta_{2m}}) \varphi^{2n}_{2m} = \tilde{\varphi}^{2n}_{2m+1} - \frac{r - e^{i\eta_m}}{t} \tilde{\varphi}^{2n}_{2m}. \]

While combining (5.1) for $\tilde{\varphi}^{2m}$ and the fact that $\varphi^{2m}$ satisfy the boundary condition at $2m$, we obtain
\[ \tilde{\varphi}^{2m}_{2m+1} = \frac{r - e^{i\eta_m}}{t} \tilde{\varphi}^{2m}_{2m}. \]

Inserting these results in (5.9), we conclude that
\[ [(U^{[2n,2m]}_{\eta_n,\eta_m} - z) G^{[2n,2m]}_{\eta_n,\eta_m} (., 2m)](2m) = 1. \]

Therefore, we have $[(U^{[2n,2m]}_{\eta_n,\eta_m} - z) G^{[2n,2m]}_{\eta_n,\eta_m} (., l)](2m) = \delta_{l,2m}$.

Combining parts (a), (b) and (c) it follows that
\[ [(U^{[2n,2m]}_{\varphi,\eta_n,\eta_m} - z) G^{[2n,2m]}_{\varphi,\eta_n,\eta_m} (., l)](k) = \delta_{kl}. \]

Thus ending the proof in the case $[a, b] = [2n, 2m]$.

The other three cases are proven using similar procedure. □
CHAPTER 6
BOUND ON FRACTIONAL MOMENTS OF GREEN’S FUNCTION

Estimates of fractional powers of Green’s function have played a central role in localization proofs for the self adjoint Anderson model, see for example [22], [4], [3]. We start our analysis of fractional moments of Green’s function of the unitary Anderson model by establishing an upper bound on their expectations that is uniform in the spectral parameter. This bound will be used later in proving exponential decay of such fractional moments.

First we state our result;

**Theorem 6.1.** Assume that \( \{\theta_k^\omega\}_{k \in \mathbb{Z}} \) are i.i.d. with probability measure \( d\mu(\theta) = \tau(\theta)d\theta \), where \( \tau \in L^\infty(\mathbb{T}) \). For \( s \in (0, 1) \) and \( 0 < \epsilon < 1 \), there exists \( 0 < \tilde{C}(s, \epsilon) < \infty \) such that

\[
E[|G_z^\omega(k, l)|^s] \leq \tilde{C}(s, \epsilon),
\]

for all \( z \in \{ x \in \mathbb{C} \setminus \{0\} : 1 - \epsilon < |x| < \infty, |x| \neq 1 \} \) and all \( k, l \in \mathbb{Z} \).

Since \( (U_\omega - z)^{-1} = \frac{1}{2\pi i} [(U_\omega + z)(U_\omega - z)^{-1} - I] \), it easy to see that there exists \( 0 < C(s, \epsilon) < \infty \) such that

\[
E[|\langle e_k, (U_\omega + z)(U_\omega - z)^{-1}e_l \rangle|^s] \leq C(s, \epsilon),
\]

Therefore, it suffices to prove the existence of \( 0 < C < \infty \) for which

\[
E[|\langle e_k, (U_\omega + z)(U_\omega - z)^{-1}e_l \rangle|^s] \leq C,
\]

for all \( z \in \{ x \in \mathbb{C} \setminus \{0\} : |x| \neq 1 \} \) and all \( k, l \in \mathbb{Z} \). We use the method of finite rank perturbation in order to establish such bound on the modified resolvent.
6.1. Finite Rank Perturbations

For \( k \neq l \), let \( A = \{k, l\} \subset \mathbb{Z} \), \( \alpha = \frac{1}{2}(\theta_k^\omega + \theta_l^\omega) \), \( \beta = \frac{1}{2}(\theta_k^\omega - \theta_l^\omega) \) and define

\[
\eta_j = \begin{cases} 
\alpha & j \in A \\
0 & j \notin A 
\end{cases}, \\
\xi_j = \begin{cases} 
\beta & j = k \\
-\beta & j = l \\
0 & j \notin A 
\end{cases}, \\
\hat{\theta}_j^\omega = \begin{cases} 
0 & j \in A \\
\theta_j^\omega & j \notin A 
\end{cases}.
\]

Next, let

\[
D_\alpha = \text{diag}\{e^{-i\eta_j}\}_{j \in \mathbb{Z}}, \\
D_\beta = \text{diag}\{e^{-i\xi_j}\}_{j \in \mathbb{Z}}, \\
\hat{D} = \text{diag}\{e^{-i\hat{\theta}_j^\omega}\}_{j \in \mathbb{Z}}.
\]

Using these definitions we can write

\[
(6.3) \quad U_\omega = D_\alpha V_\omega,
\]

with the unitary operator \( V_\omega = D_\beta \hat{D}S \) independent of \( \alpha \). Also, it is easy to see that \( U - V \) is a finite rank operator. In what follows we explore the relation between the modified resolvents of \( U_\omega \) and \( V_\omega \).

Let \( P_A \) be the orthogonal projection onto the span of \( \{V_\omega^{-1}e_j : j \in A\} \). Using that \( \{V_\omega^{-1}e_j : j \in \mathbb{Z}\} \) is an orthonormal basis of \( l^2(\mathbb{Z}) \), simple calculations show that \( (U_\omega - V_\omega)(I - P_A) = 0 \) and \( V_\omega^{-1}U_\omega = e^{-i\alpha}I \) on range \( P_A \). Therefore,

\[
(6.4) \quad U_\omega = V_\omega(I - P_A) + e^{-i\alpha}V_\omega P_A.
\]
For \( z \in \{ x \in \mathbb{C} \setminus \{0\} : |x| \neq 1 \} \), let \( F_z = P_A(U_\omega + z)(U_\omega - z)^{-1}P_A \) while \( \hat{F}_z = P_A(V_\omega + z)(V_\omega - z)^{-1}P_A \). Using this definition, we see that

\[
\hat{F}_z + \hat{F}_z^* = P_A(2I - 2|z|^2)(V_\omega - z)^{-1}[(V_\omega - z)^{-1}]^* P_A.
\]

The invertibility of this operator, on range \( P_A \), implies that \( \hat{F}_z + \hat{F}_z^* < 0 \) for \( |z| > 1 \). Therefore, \( -i\hat{F}_z \) is a dissipative operator. Similarly, \( -i\hat{F}_z^{-1} \) is also a dissipative operator. In the case \( |z| < 1 \), we have that \( i\hat{F}_z, i\hat{F}_z^{-1} \) are dissipative.

Next we explore the relation between \( \hat{F}_z \) and \( F_z \). Following [35], we use the fact that

\[
(x + z)(x - z)^{-1} = 1 + 2z(x - z)^{-1}
\]

along with (6.4), to obtain

\[
F_z - \hat{F}_z = -2zP_A(V_\omega - z)^{-1}V_\omega P_A(e^{-i\alpha} - 1)P_A(U_\omega - z)^{-1}P_A.
\]

Since \((V_\omega - z)^{-1}V_\omega = \frac{1}{2}(1 + (V_\omega + z)(V_\omega - z)^{-1}) \) and \( 2z(U_\omega - z)^{-1} = (U_\omega + z)(U_\omega - z)^{-1} - 1 \), it follows that

\[
F_z - \hat{F}_z = \frac{1}{2}(1 + \hat{F}_z)(e^{-i\alpha} - 1)(1 - F_z).
\]

Therefore, a straightforward calculation shows that

\[
F_z = \frac{-(1 - e^{-i\alpha}) + (1 + e^{-i\alpha})\hat{F}_z}{1 + e^{-i\alpha} - (1 - e^{-i\alpha})\hat{F}_z}.
\]

For \( \alpha \notin \{0, \pi\} \), let \( m(\alpha) = i \frac{1 + e^{-i\alpha}}{1 - e^{-i\alpha}} \in \mathbb{R} \). Thus, we have

\[
F_z = \frac{-i}{-i\hat{F}_z + m(\alpha)} = \frac{i}{-i\hat{F}_z^{-1} - m^{-1}(\alpha)}.
\]

As a first step to establishing a uniform upper bound on the fractional moments of the modified resolvent, we prove the following bound for \( F_z \);

**Lemma 6.1.** For \( s \in (0, 1) \), there exists a constant \( 0 < C(s) < \infty \) such that

\[
\int_0^{2\pi} d\alpha |\langle f, F_z g \rangle|^s \leq C,
\]

for all unit vectors \( f, g \in l^2(\mathbb{Z}) \) and \( z \in \{ x \in \mathbb{C} \setminus \{0\} : |x| \neq 1 \} \).
Proof. First assume that $|z| > 1$. In light of equation (6.5) and the fact that $||f|| = ||g|| = 1$, we see that
\[
\int_0^{2\pi} \, d\alpha |\langle f, F_z g \rangle|^s \leq C_1(s) \left\{ \int_0^{2\pi} \, d\alpha |\langle f, \frac{1}{-i\hat{F}_z + m(\alpha)} g \rangle|^s + \int_0^{2\pi} \, d\alpha |\langle f, \frac{1}{-i\hat{F}_z^{-1} - m^{-1}(\alpha)} g \rangle|^s \right\}
\]
(6.6)
\[
\leq C_1(s) \left\{ \int_0^{2\pi} \, d\alpha ||P_A \frac{1}{-i\hat{F}_z + m(\alpha)} P_A||^s + \int_0^{2\pi} \, d\alpha ||P_A \frac{1}{-i\hat{F}_z^{-1} - m^{-1}(\alpha)} P_A||^s \right\}.
\]
Recalling the definition of $m(\alpha)$ and making a change of variables by setting $x = m(\alpha)$, it is not difficult to see that
\[
\int_0^{2\pi} \, d\alpha ||P_A \frac{1}{-i\hat{F}_z + m(\alpha)} P_A||^s = \lim_{\epsilon \to 0} \int_0^{2\pi - \epsilon} \, d\alpha ||P_A \frac{1}{-i\hat{F}_z + m(\alpha)} P_A||^s
\]
\[
= \lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{x^2 + 1} ||P_A \frac{1}{-i\hat{F}_z + x} P_A||^s
\]
\[
= 2 \sum_{n \in \mathbb{Z}} \int_{n}^{n+1} \frac{dx}{x^2 + 1} ||P_A \frac{1}{-i\hat{F}_z + x} P_A||^s
\]
(6.7)
\[
\leq 2 \sum_{n \in \mathbb{Z}} \frac{1}{(|n| - 1)^2 + 1} \int_{n}^{n+1} \frac{dx}{x^2 + 1} ||P_A \frac{1}{-i\hat{F}_z + x} P_A||^s,
\]
where the last inequality follows from the fact that for all $n \in \mathbb{Z}$,
\[
\min_{x \in [n, n+1]} (x^2 + 1) \geq (|n| - 1)^2 + 1.
\]
Since $-i\hat{F}_z$ is a dissipative finite rank operator, Lemma (3.1) of [2] asserts the existence of a constant $0 < C_3 < \infty$ independent of $P_A, \hat{F}_z$ such that
\[
\left| \{ x \in [n, n + 1] : ||P_A \frac{1}{-i\hat{F}_z + x} P_A||_{HS} > t \} \right| \leq C_3 \frac{1}{t}.
\]
Using the “layer-cake” representation, we conclude that
\[
\int_{n}^{n+1} \frac{dx}{x^2 + 1} ||P_A \frac{1}{-i\hat{F}_z + x} P_A||_{HS} = \int_0^\infty dt \left| \{ x \in [n, n + 1] : ||P_A \frac{1}{-i\hat{F}_z + x} P_A||_{HS} > t^{1/s} \} \right|
\]
\[
\leq C_3 (1 + \frac{s}{s + 1}),
\]
where we have used that $\left| \{ x \in [n, n + 1] : ||P_A \frac{1}{-i\hat{F}_z + x} P_A||_{HS} > t^{1/s} \} \right| \leq \max\{1, C_3 t^{-1/s}\}$.
Inserting this result in equation (6.7) and noting that $P_A$ is a finite rank operator, it follows
that
\[ \int_0^{2\pi} d\alpha ||P_A \frac{1}{-i\hat{F}_z + m(\alpha)} P_A||^s \leq 2\tilde{C}_3(1 + \frac{s}{s+1}) \sum_{n\in\mathbb{Z}} \frac{1}{(|n|-1)^2 + 1}. \]

Finally noting that \( \sum_{n\in\mathbb{Z}} \frac{1}{(|n|-1)^2 + 1} \) is a bounded series allows us to conclude that
\[ \int_0^{2\pi} d\alpha ||P_A \frac{1}{-i\hat{F}_z + m(\alpha)} P_A||^s \leq C_4, \]
with \( C_4 \) depending only on \( s \).

Next we show the existence of a constant \( 0 < C_5(s) < \infty \) such that
\[ \int_0^{2\pi} d\alpha ||P_A \frac{1}{-i\hat{F}_z^{-1} - m^{-1}(\alpha)} P_A||^s \leq C_5(s). \]

Since \( m^{-1}(\alpha) \) has a singularity at \( \alpha = \pi \), then
\[ \int_0^{2\pi} d\alpha ||P_A \frac{1}{-i\hat{F}_z^{-1} - m^{-1}(\alpha)} P_A||^s = \lim_{\epsilon \to 0} \left( \int_0^{\pi - \epsilon} + \int_{\pi + \epsilon}^{2\pi} \right) d\alpha ||P_A \frac{1}{-i\hat{F}_z^{-1} - m^{-1}(\alpha)} P_A||^s. \]

As before we make a change of variables by letting \( y = -m^{-1}(\alpha) \) and use the fact that \( -i\hat{F}_z^{-1} \) is a dissipative operator and the assertion of Lemma (3.1) of [2] to obtain
\[ \int_0^{2\pi} d\alpha ||P_A \frac{1}{-i\hat{F}_z^{-1} - m^{-1}(\alpha)} P_A||^s \leq 2 \lim_{R \to \infty} \int_R^R dy \frac{1}{y^2 + 1} ||P_A \frac{1}{-i\hat{F}_z^{-1} + y} P_A||^s \leq C_5(s). \]

These proven bounds along with (6.6) conclude the proof for that case \(|z| > 1\). The case \(|z| < 1\) is done similarly with \(-i\hat{F}_z\) and \(-i\hat{F}_z^{-1}\) replaced with \(i\hat{F}_z\), \(i\hat{F}_z^{-1}\) respectively. \( \square \)

6.2. Proof of The Main Theorem

We are now ready to prove the existence of a uniform upper bound on the expectation of fractional powers of Green’s function.

Proof of Theorem 6.1. As discussed earlier, it suffices to show that there exists \( 0 < C < \infty \) for which
\[ \mathbb{E}[|\langle e_k, (U_\omega + z)(U_\omega - z)^{-1} e_l \rangle|^s] \leq C, \]
for all \( z \in \{x \in \mathbb{C}\setminus\{0\} : |x| \neq 1\} \) and all \( k, l \in \mathbb{Z} \).
Since $U_\omega$ is unitary and $V_\omega^{-1}U_\omega = e^{-i\alpha}I$ on span$\{V_\omega^{-1}e_j : j \in \{k, l\}\}$, we have

$$\langle e_k, (U_\omega + z)(U_\omega - z)^{-1}e_l \rangle = \langle e_k, (U_\omega + z)(U_\omega - z)^{-1}VV_\omega^{-1}e_l \rangle$$

$$= \langle e_k, (U_\omega + z)(U_\omega - z)^{-1}e^{i\alpha}U_\omega V_\omega^{-1}e_l \rangle$$

$$= e^{i\alpha} \langle V_\omega V_\omega^{-1}e_k, U_\omega(U_\omega + z)(U_\omega - z)^{-1}V_\omega^{-1}e_l \rangle$$

$$= \langle V_\omega^{-1}e_k, F_zV_\omega^{-1}e_l \rangle,$$

where the last equality follows from the definitions of $F_z$ and $P_A$.

Therefore, it is easy to see that

$$\mathbb{E}[\|\langle e_k, (U_\omega + z)(U_\omega - z)^{-1}e_l \rangle\|^s] = \hat{\mathbb{E}}\left[ \int_0^{2\pi} d\mu(\theta^e_k) \int_0^{2\pi} d\mu(\theta^e_l) |\langle e_k, (U_\omega + z)(U_\omega - z)^{-1}e_l \rangle|^s \right]$$

$$\leq ||\tau||_s^s \hat{\mathbb{E}}\left[ \int_0^{2\pi} \int_0^{2\pi} |\langle V_\omega^{-1}e_k, F_zV_\omega^{-1}e_l \rangle|^s d\theta^e_k d\theta^e_l \right],$$

(6.8)

where $\hat{\mathbb{E}}$ is the expectation with respect the random variables $\{\theta^e_j\}, j \notin \{j, k\}$. Making a change of variables from $\{\theta_k, \theta_l\}$ to $\{\alpha, \beta\} = \left(\frac{1}{2}(\theta^e_k + \theta^e_l), \frac{1}{2}(\theta^e_k - \theta^e_l)\right)$, we see that

$$\int_0^{2\pi} \int_0^{2\pi} |\langle V_\omega^{-1}e_k, F_zV_\omega^{-1}e_l \rangle|^s d(\theta_k)d(\theta_l) \leq 2 \int \int_R |\langle V_\omega^{-1}e_k, F_zV_\omega^{-1}e_l \rangle|^s d\alpha d\beta,$$

where $R$ is a rectangular area given by $0 \leq \alpha \leq 2\pi, -\pi \leq \beta \leq \pi$. Thus, using the result of Lemma 6.1, the later integration can be estimated by

$$\int_0^{2\pi} \int_0^{2\pi} |\langle V_\omega^{-1}e_k, F_zV_\omega^{-1}e_l \rangle|^s d(\theta_k)d(\theta_l) \leq 2 \int_{-\pi}^{\pi} \int_0^{2\pi} d\beta \int_0^{2\pi} d\alpha |\langle V_\omega^{-1}e_k, F_zV_\omega^{-1}e_l \rangle|^s$$

$$\leq 4\pi C.$$

Combining this with (6.2) and (6.8) concludes the proof for $k \neq l$. The case $k = l$ is similar but easier, since no change of variable is needed and one directly uses Lemma 6.1 with $\alpha = \theta_k$. □

**Corollary 6.1.** Let $G_z^{[a, b]}$ be defined by (5.7), (2.21). For $s \in (0, 1)$ and $0 < \epsilon < 1$, there exists $0 < \tilde{C}(s, \epsilon) < \infty$ such that

$$\mathbb{E}[|G_z^{[a, b]}(k, l)|^s] \leq \tilde{C}(s, \epsilon),$$

(6.9)

for all $z \in \{x \in \mathbb{C} : 1 - \epsilon < |x| < \infty, |x| \neq 1\}$ and all $k, l \in [a, b]$. 

Proof. The proof follows exactly as the proof of Theorem 6.1, with $U_\omega$ and $D_\omega$ replaced with $U_\omega^{[a,b]}$ and $D_\omega^{[a,b]}$ respectively. \qed
CHAPTER 7

EXPONENTIAL DECAY OF FRACTIONAL MOMENTS

Once uniform boundedness of the expectation of fractional powers of Green’s function is established, the next step is proving that such expectation decays exponentially as the distance between $k, l$ increases. For the unitary Anderson model such estimate was proven in [27], in all dimensions, under the assumption that $t$ is small. This proof follows the technique used in [4], [3] and [22] to prove such estimates for the self-adjoint Anderson model in the high disorder regime. For the one-dimensional self-adjoint Anderson model such decay was proven in [31], without any disorder assumptions, using strictly one-dimensional tools and the specific form of the discrete Laplacian. In order to prove this decay estimate for all $0 < t < 1$ for the unitary model, we utilize the results concerning positivity of the Lyapunov exponent as well as the Green’s function formula developed earlier in the manuscript. Indeed, using these results we are able to prove the following:

**Theorem 7.1.** Assume that $\{\theta_k^\omega\}_{k \in \mathbb{Z}}$ are i.i.d. with probability measure $d\mu(\theta) = \tau(\theta)d\theta$, where $\tau \in L^\infty(\mathbb{T})$ and $0 < \epsilon < 1/2$. There exist $s \in (0, 1/8)$, $0 < C < \infty$ and $\alpha > 0$ such that

$$
\mathbb{E}[|G_\omega^z(k,l)|^s] \leq Ce^{-\alpha|k-l|},
$$

for all $z \in \{x \in \mathbb{C} : 0 < |x| - 1| < \epsilon\}$, and $k, l \in \mathbb{Z}$.

for $-\infty \leq a < b \leq \infty$, a similar estimate holds for $G^{[a,b]}_z$, defined by (5.7), where $U^{[a,b]}_\omega$ is defined by (2.21), and $\eta_a = \eta_b = 0$ when either $a$ or $b$ is finite.

Before we prove the Theorem we will show that proving the bound (7.1) for an element $(k, l)$ of $(U^{[a,b]}_\omega - z)^{-1}$ can be reduced to proving the same bound for even elements $(2n, 2m)$ of the resolvent of the finite volume operator $U^{[2n,2m]}_\omega$ at $z$, i.e. $G^{[2n,2m]}_z(2n,2m)$. 

This reduction is done in two steps, first we show that it suffices to deal with even elements \( G_{2}^{[a,b]}(2n, 2m) \), then we reduce the problem to the required finite volume, i.e. to \( G_{2}^{[2n,2m]}(2n, 2m) \).

### 7.1. Reduction to Even Elements

In this section we show that the expectation of a fractional moment of any element \( G_{2}^{[a,b]}(k, l) \) can be reduced to the expectation of fractional moments of even elements of \( G_{2}^{[a,b]} \).

**Lemma 7.1.** Let \( s \in (0, 1/4) \), \( 0 < \epsilon < 1/2 \), and \( k, l \in [a + 2, b - 2] \) such that \( |k - l| > 4 \). Then there exists \( 0 < \kappa(r, s, \mu, \epsilon) < \infty \) such that

\[
(\mathbb{E}[|G_{2}^{[a,b]}(k, l)|^{s}])^{2} \leq \kappa(r, s, \mu, \epsilon) \sum_{i,j=0}^{1} (\mathbb{E}[|G_{2}^{[a,b]}(2n + 2i, 2m - 2j)|^{4s}])^{1/2},
\]

for all \( z \in \{ x \in \mathbb{C} : 0 < |x| - 1 < \epsilon \} \), and \( n, m \in \mathbb{Z} \) such that \( k \in \{ 2n, 2n + 1 \} \), \( l \in \{ 2m, 2m + 1 \} \).

**Proof.** Using the definition of \( n, m \) and that \( |k - l| > 4 \) one has \( |n - m| \geq 2 \). Let \( \varphi^- \) denotes the solution of \( (U - z)\psi = 0 \) that satisfies the zero boundary condition at \( a \), while \( \varphi^+ \) is the one satisfying the zero boundary condition at \( b \). Since \( \varphi^\pm \), defined by (5.1), satisfies (5.3) and (5.4), a straightforward calculation shows that

\[
\tilde{\varphi}_{2m+1}^{\pm} = \frac{1}{r(t(e^{i\theta_{2m+1}} - 1/z))} \{ [r^2(e^{i\theta_{2m+1}} + e^{i\theta_{2m-1}} - 1/z) - z e^{i(\theta_{2m+1} + \theta_{2m-1})}] \tilde{\varphi}_{2m}^{\pm} - t^2 e^{i\theta_{2m}} \tilde{\varphi}_{2m-2}^{\pm} \}.
\]

Using Theorem 5.1 along with (5.3) it follows that for \( k \notin \{ 2m - 1, 2m \} \)

\[
G_{2}^{[a,b]}(k, 2m + 1) = \frac{e^{i\theta_{2m+1}}}{r(t(e^{i\theta_{2m-1}} - 1/z))} \{ [r^2(1 + e^{-i(\theta_{2m-1})} - \frac{e^{-i\theta_{2m}}}{z}) - z e^{i\theta_{2m-1}}] G_{2}^{[a,b]}(k, 2m) - t^2 e^{i(\theta_{2m-1} - \theta_{2m-2})} G_{2}^{[a,b]}(k, 2m - 2) \}.
\]

Therefore, we have

\[
|G_{2}^{[a,b]}(k, 2m + 1)| \leq \frac{1}{r(t(e^{i\theta_{2m-1}} - 1/z))} \{ [3r^2 + 2]|G_{2}^{[a,b]}(k, 2m)| + t^2|G_{2}^{[a,b]}(k, 2m - 2)| \}.
\]
By Hölder’s inequality,
\[
(\mathbb{E}[|G_z^{[a,b]}(k, 2m + 1)|^s])^2 \leq \mathbb{E}\left[\frac{1}{|e^{i\theta} - \beta|^s}\right]\{C_1(r, s)\mathbb{E}[|G_z^{[a,b]}(k, 2m)|^{2s}] + C_2(r, s)\mathbb{E}[|G_z^{[a,b]}(k, 2m - 2)|^{2s}]\}.
\]

Since Lemma A.2 asserts that for all \(0 < s < 1\), there exists \(0 < C_\mu(s) < \infty\) such that for all \(\beta \in \mathbb{C}\)
\[
\int_{\mathbb{T}} d\mu(\theta) \frac{1}{|e^{i\theta} - \beta|^s} \leq C_\mu(s),
\]

(7.2)
it follows that for \(s \in (0, 1/4)\)
\[
(\mathbb{E}[|G_z^{[a,b]}(k, 2m + 1)|^s])^2 \leq C_\mu^{(1)}(s, r)\{\mathbb{E}[|G_z^{[a,b]}(k, 2m)|^{2s}] + \mathbb{E}[|G_z^{[a,b]}(k, 2m - 2)|^{2s}]\},
\]

(7.3)
with \(C_\mu^{(1)}(s, r) = C_\mu(s) \max\{C_1(r, s), C_2(r, s)\}\).

Similarly, using (2.6) and (2.7) we obtain
\[
\Phi_{2n+1}^\pm = \frac{-t}{r(e^{i\theta} - \beta)^{2n+1}} \left\{\frac{1}{z} \Phi_{2n+2}^\pm + e^{i\theta} \Phi_{2n}^\pm\right\}.
\]

Thus, for \(l \notin \{2n, 2n + 1\}, s \in (0, 1/2)\) and all \(z \in \{x \in \mathbb{C} : ||x| - 1| < \epsilon, |x| \neq 1\}, 0 < \epsilon < 1/2\)
\[
(\mathbb{E}[|G_z^{[a,b]}(2n + 1, l)|^s])^2 \leq C_\mu^{(2)}(s, r)\{\mathbb{E}[|G_z^{[a,b]}(2n + 2, l)|^{2s}] + \mathbb{E}[|G_z^{[a,b]}(2n, l)|^{2s}]\}.
\]

(7.4)
Therefore, it readily follows from (7.3) and (7.4) that for \(|n - m| \notin \{0, 1\}\) and all \(s \in (0, 1/4)\)
\[
(\mathbb{E}[|G_z^{[a,b]}(2n + 1, 2m + 1)|^s])^2 \leq \tilde{K}_\mu^{(1)}(s, r)\left\{(\mathbb{E}[|G_z^{[a,b]}(2n + 2, 2m)|^{4s})\right\}^{1/2}
\]
\[
+ (\mathbb{E}[|G_z^{[a,b]}(2n, 2m)|^{4s})\right\}^{1/2}
\]
\[
+ (\mathbb{E}[|G_z^{[a,b]}(2n + 2, 2m - 2)|^{4s})\right\}^{1/2}
\]
\[
+ (\mathbb{E}[|G_z^{[a,b]}(2n + 2, 2m - 2)|^{4s})\right\}^{1/2}\}
\]

(7.5)
Finally using Jensen’s inequality, it follows that for \(s \in (0, 1/4)\) and for all \(k, l\)
\[
\mathbb{E}[|G_z^{[a,b]}(k, l)|^{2s}] \leq \left\{(\mathbb{E}[|G_z^{[a,b]}(k, l)|^{4s})\right\}^{1/2}.
\]

Thus, the conclusion of the Lemma follows from (7.3), (7.4) and (7.5). \(\square\)
7.2. Reduction to the Appropriate Finite Volume

In order to simplify the notation, let \( |n, m| \) denote the interval \( [\min\{n, m\}, \max\{n, m\}] \).

In what follows we show that the expectation of fractional moments of \( G_z^{[a, b]}(2n, 2m) \), for arbitrary \( a, b \), can be reduced to that of \( G_z^{[2n, 2m]}(2n, 2m) \).

**Lemma 7.2.** For \( s \in (0, 1/2) \), \( 0 < \epsilon < 1/2 \), we have

\[
(\mathbb{E}[|G_z^{[a, b]}(k, l)|^s])^2 \leq C(t, s, \epsilon)\mathbb{E}[|G_z^{[k, l]}(k, l)|^{2s}],
\]

for all \( z \in \{x \in \mathbb{C} : 0 < |x| - 1| < \epsilon \} \) and all even \( k, l \in [a + 2, b - 2] \) with \( |k - l| > 4 \).

**Proof.** Since we are dealing with even \( k, l \) with \( |k - l| \geq 4 \), then there exist \( m, n \in \mathbb{Z} \) such that \( k = 2n, l = 2m \). The proof is naturally divided into two parts:

**Part I:** We start with the case \( m \geq n + 2 \). Using the definition of \( U_\omega^{[x, y]} \), (2.21), we see

\[
U_\omega^{[a, b]} = U_\omega^{[a, 2n - 1]} \oplus U_\omega^{[2n, b]} + \Gamma_n^e,
\]

where \( \Gamma_n^e \) is given by

\[
\Gamma_n^e(k, l) = \begin{cases} 
(rt - t)e^{-i\theta_{2n-2}}, & k = 2n - 2, l = 2n - 1 \\
-t^2e^{-i\theta_{2n-2}}, & k = 2n - 2, l = 2n \\
(r^2 - r)e^{-i\theta_{2n-1}}, & k = 2n - 1, l = 2n - 1 \\
-rte^{-i\theta_{2n-1}}, & k = 2n - 1, l = 2n \\
re^{-i\theta_{2n}}, & k = 2n, l = 2n - 1 \\
(r^2 - r)e^{-i\theta_{2n}}, & k = 2n, l = 2n \\
-t^2e^{-i\theta_{2n+1}}, & k = 2n + 1, l = 2n - 1 \\
(-rt + t)e^{-i\theta_{2n+1}}, & k = 2n + 1, l = 2n \\
0, & \text{otherwise.}
\end{cases}
\]

Denote \( G_z^m = G_z^{[a, 2n - 1]} \oplus G_z^{[2n, b]} \). By the first resolvent identity, we have

\[
G_z^{[a, b]} - G_z^m = -G_z^{[a, b]}\Gamma_n^e G_z^m.
\]
Therefore, it follows for all \( m \geq n + 2 \) that

\[
G_z^{[a,b]}(2n, 2m) = \{1 + t^2 e^{-i\theta_{2n-2}}G_z^{[a,b]}(2n, 2n - 2) + \text{rt}e^{-i\theta_{2n-1}}G_z^{[a,b]}(2n, 2n - 1) \\
- (r^2 - r)e^{-i\theta_{2n}}G_z^{[a,b]}(2n, 2n) - (t - \text{rt})e^{-i\theta_{2n+1}}G_z^{[a,b]}(2n, 2n + 1)\}G_z^{[2n,b]}(2n, 2m).
\]

(7.8)

Similarly, one can rewrite \( U_\omega^{[2n,2m]} \) as

\[
U_\omega^{[2n,b]} = U_\omega^{[2n,2m]} \oplus U_\omega^{[2m+1,b]} + \Gamma^m_0,
\]

where \( \Gamma^m_0 \) is given by

\[
\Gamma^m_0(k, l) = \begin{cases} 
(rt - t)e^{-i\theta_{2m}}, & k = 2m, l = 2m - 1 \\
(r^2 - r)e^{-i\theta_{2m}}, & k = 2m, l = 2m \\
\text{rt}e^{-i\theta_{2m}}, & k = 2m, l = 2m + 1 \\
-t^2 e^{-i\theta_{2m}}, & k = 2m, l = 2m + 2 \\
-t^2 e^{-i\theta_{2m+1}}, & k = 2m + 1, l = 2m - 1 \\
-\text{rt}e^{-i\theta_{2m+1}}, & k = 2m + 1, l = 2m \\
(r^2 - r)e^{-i\theta_{2m+1}}, & k = 2m + 1, l = 2m + 1 \\
(-rt + t)e^{-i\theta_{2m+1}}, & k = 2m + 1, l = 2m + 2 \\
0, & \text{otherwise.}
\end{cases}
\]

(7.9)

Now if we let \( G_z^m = G_z^{[2n,2m]} \oplus G_z^{[2m+1,b]} \), again we see that

\[
G_z^{[2n,b]} - G_z^m = -G_z^m \Gamma^m_0 G_z^{[2n,b]}.
\]

Thus, for all \( m \geq n + 2 \)

\[
G_z^{[2n,b]}(2n, 2m) = \{1 - e^{-i\theta_{2m}}[(rt - t)G_z^{[2n,b]}(2m - 1, 2m) + (r^2 - r)G_z^{[2n,b]}(2m, 2m) \\
- \text{rt}G_z^{[2n,\infty]}(2m + 1, 2m) - t^2 G_z^{[2n,b]}(2m + 2, 2m)]\}G_z^{[2n,2m]}(2n, 2m).
\]
Inserting this in (7.8), we obtain the following expression for $G_z^{[a,b]}(2n, 2m)$

$$G_z^{[a,b]}(2n, 2m) = \left\{ (1 + t^2e^{-i\theta_{2m+2}}G_z^{[a,b]}(2n, 2n-2) + rte^{-i\theta_{2m+1}}G_z^{[a,b]}(2n, 2n-1) - (r^2 - r)e^{-i\theta_{2m}}G_z^{[a,b]}(2n, 2n) - (t - rt)e^{-i\theta_{2m+1}}G_z^{[a,b]}(2n, 2n+1) \right\} G_z^{[a,b]}(2n, 2m),$$

for all $m \geq n + 2$.

Using Hölder’s inequality, the assertions of Theorem 6.1 and Corollary 6.1, equation (7.10) gives for all $0 < s < 1/2$, $0 < \epsilon < 1$,  

$$(\mathbb{E}[|G_z^{[a,b]}(2n, 2m)|^s])^2 \leq C_{\mu}(t, s, \epsilon) \mathbb{E}[|G_z^{[2m,2m]}(2n, 2m)|^{2s}],$$

yielding the required result for the case $m \geq n + 2$.

**Part II:** Next we assume that $n \geq m + 2$. Following the same procedure as part I, with roles of $m, n$ interchanged, the first resolvent identity gives 

$$G_z^{[a,b]} - G_z^m = -G_z^m \Gamma^e_m G_z^{[a,b]}.$$

Therefore, 

$$G_z^{[a,b]}(2n, 2m) = \left\{ 1 - [t(1-r)e^{-i\theta_{2m+1}} - rte^{-i\theta_{2m}}]G_z^{[a,b]}(2m - 1, 2m) \right\} G_z^{[2m,b]}(2n, 2m),$$

where we have used that $G_z^{[2m,b]}(2n, 2m + 1) = \frac{r - 1}{t}G_z^{[2m,b]}(2n, 2m)$.

Similarly, we have 

$$G_z^{[2m,b]}(2n, 2m) = \left\{ 1 + (r - 1)(z - 2re^{-i\theta_{2m}})G_z^{[2m,b]}(2n, 2n) \right\} G_z^{[2m,2m]}(2n, 2m).$$

Using the same arguments as part I, combining those two equations again implies that 

$$(\mathbb{E}[|G_z^{[a,b]}(2n, 2m)|^s])^2 \leq C_{\mu}(t, s, \epsilon) \mathbb{E}[|G_z^{[2m,2m]}(2n, 2m)|^{2s}],$$
for all \( z \in \{ x \in \mathbb{C} : 0 < |x| - 1| < \epsilon \} \) and all \( n, m \) such that \( n \geq m + 2 \). This concludes the proof. \( \square \)

### 7.3. Exponential Decay of the Reduced Case

Now that we have reduced the problem to dealing with fractional moments of elements of the form \( G_z^{[2n,2m]}(2n,2m) \), we show that expectations of such objects decay exponentially, in particular we show;

**Lemma 7.3.** Assume that \( \{ \theta_k^x \}_{k \in \mathbb{Z}} \) are i.i.d. with probability measure \( d\mu(\theta) = \tau(\theta)d\theta \), where \( \tau \in L^\infty(\mathbb{T}) \) and \( 0 < \epsilon < 1/2 \). There exist \( s_0 \in (0,1) \), \( 0 < C_1 < \infty \), \( \alpha_1 > 0 \) such that

\[
E[|G_z^{[2n,2m]}(2n,2m)|^s] \leq C_1 e^{-\alpha_1(m-n)},
\]

for all \( z \in \{ x \in \mathbb{C} : 0 < |x| - 1| < \epsilon \} \), and \( m, n \in \mathbb{Z} \) such that \( |m-n| \geq 2 \).

**Proof.** As in the proof of Lemma 7.2, we will consider two separate situations:

**Part I:** For \( m \geq n+2 \), let \( \varphi^{2n}(\varphi^{2m}) \) be two solutions that satisfy the 0-left(right)boundary conditions at \( 2n \) (2m) respectively, such that \( \varphi^{2n}_{2n} = 1 \) and \( \varphi^{2m}_{2m} = 1 \). Using (5.8), we have

\[
G_z^{[2n,2m]}(2n,2m) = e^{i\theta_{2m}} \frac{\varphi^{2n}_{2n}}{\varphi^{2m}_{2m+1} - \varphi^{2n}_{2m} \varphi^{2m}_{2m+1}} \varphi^{2m}_{2m}.
\]

Using the definition of \( \varphi^{2m} \), (5.1), and the fact that \( \varphi^{2m} \) satisfies the 0-boundary condition at \( 2m \), it follows that

\[
\varphi^{2m}_{2m} = tze^{i\theta_{2m}},
\]

\[
\varphi^{2m}_{2m+1} = (r - 1)z e^{i\theta_{2m}}.
\]

Therefore,

\[
G_z^{[2n,2m]}(2n,2m) = \frac{te^{i\theta_{2m}}}{t\varphi^{2n}_{2m+1} + (1-r)\varphi^{2n}_{2m}} = \frac{e^{i\theta_{2m}}}{t\varphi^{2n}_{2m-1} + (r - z e^{i\theta_{2m}})\varphi^{2n}_{2m}},
\]

where the last equality follows from the definition of \( \varphi^{2n} \) (5.1).
Now, for \( s \in (0, 1) \) the expectation of the s-moment of \( G_{z}^{[2n,2m]}(2n, 2m) \) is given by

\[
\mathbb{E}[(G_{z}^{[2n,2m]}(2n, 2m))^s] = \mathbb{E}[\int_0^{2\pi} d\mu(\theta_{2m}) \frac{1}{|t\varphi_{2m-1}^{2n} + (r - ze^{i\theta_{2m}^s})\varphi_{2m}^{2n}|^s}],
\]

where \( \mathbb{E} \) is the expectation with respect to the random variables \( \{\theta_k\}_{k \in \mathbb{Z}\setminus\{2m\}} \). In order to estimate this expectation, we first note that both \( \varphi_{2m-1}^{2n} \) and \( \varphi_{2m}^{2n} \) depend only on the phases \( \theta_j^s \) with \( 2n - 1 \leq j \leq 2m - 1 \), in particular each is independent of \( \theta_{2m}^s \). Also, since \( \varphi_{2m}^{2n} \) is a non trivial solution, \( \varphi_{2m-1}^{2n} \) and \( \varphi_{2m}^{2n} \) can not vanish simultaneously. Therefore we have the following cases:

**Case 1:** \( \varphi_{2m}^{2n} = 0 \), using that \( \left\| \begin{pmatrix} \varphi_{2m-1}^{2n} \\ \varphi_{2m}^{2n} \end{pmatrix} \right\| \leq 2|\varphi_{2m-1}^{2n}| + 2|\varphi_{2m}^{2n}| \), it follows that

\[
\int_0^{2\pi} d\mu(\theta_{2m}) \frac{1}{|t\varphi_{2m-1}^{2n} + (r - ze^{i\theta_{2m}^s})\varphi_{2m}^{2n}|^s} \leq \left\| \tau \right\|_\infty \frac{2\pi}{|t\varphi_{2m-1}^{2n}|^s} \leq \left\| \tau \right\|_\infty \frac{2s+1\pi}{|t|^s} \left\| \begin{pmatrix} \varphi_{2m-1}^{2n} \\ \varphi_{2m}^{2n} \end{pmatrix} \right\|^{-s}.
\]

**Case 2:** \( \varphi_{2m-1}^{2n} = 0 \), in this case we have that

\[
\int_0^{2\pi} d\mu(\theta_{2m}) \frac{1}{|t\varphi_{2m-1}^{2n} + (r - ze^{i\theta_{2m}^s})\varphi_{2m}^{2n}|^s} = \left\| \begin{pmatrix} \varphi_{2m-1}^{2n} \\ \varphi_{2m}^{2n} \end{pmatrix} \right\|^{-s} 2^s \int_0^{2\pi} d\mu(\theta) \frac{1}{|e^{i\theta} - r/z|^s} \leq C_\mu(s)\left\| \begin{pmatrix} \varphi_{2m-1}^{2n} \\ \varphi_{2m}^{2n} \end{pmatrix} \right\|^{-s},
\]

where the last inequality is obtained using that \( |z| > 1/2 \) and the assertion of Lemma A.2; see equation (7.2).

**Case 3:** both \( \varphi_{2m-1}^{2n} \neq 0 \) and \( \varphi_{2m-1}^{2n} \neq 0 \). In this case

\[
\int_0^{2\pi} d\mu(\theta_{2m}) \frac{1}{|t\varphi_{2m-1}^{2n} + (r - ze^{i\theta_{2m}^s})\varphi_{2m}^{2n}|^s} = \frac{1}{|\varphi_{2m}^{2n}|^s} \int_0^{2\pi} d\mu(\theta_{2m}) \frac{1}{|t\varphi_{2m-1}^{2n} + (r - ze^{i\theta_{2m}^s})|^s}.
\]

Let \( M = \sup\{|r - ze^{i\theta_{2m}^s}| : \theta \in [0, 2\pi], 0 < |z| - 1| < \epsilon \} < \infty \) and let’s distinguish between two subcases;
Case 3a: If $t \frac{\varphi_{2m-1}}{\varphi_{2m}} > 2M$, then

$$\left| t \frac{\varphi_{2m-1}}{\varphi_{2m}} + (r - ze^{i\theta_{2m}}) \right| \geq t \left| \frac{\varphi_{2m-1}}{\varphi_{2m}} \right| - |r - ze^{i\theta_{2m}}|$$

$$\geq t \left| \frac{\varphi_{2m-1}}{\varphi_{2m}} \right| \geq \frac{t}{2} \left| \varphi_{2m-1} \right| .$$

Therefore, it follows that

$$\int_{0}^{2\pi} d\mu(\theta_{2m}) \frac{1}{|t\varphi_{2m-1} + (r - ze^{i\theta_{2m}})\varphi_{2m}|^s} \leq \frac{2^{s+1}||\tau||_{\infty}}{t^s|\varphi_{2m-1}|^s}.$$ 

Now using that $\frac{\varphi_{2n}}{\varphi_{2m}} \leq 2|\varphi_{2m-1}| + 2|\varphi_{2n}|$ and that $M > 0$, we obtain the following bound

$$\left\| \begin{pmatrix} \varphi_{2m-1} \\ \varphi_{2m} \end{pmatrix} \right\|^s \leq \left( \frac{2 + 1}{M} \right)^s |\varphi_{2m-1}|^s .$$

Using these estimates we conclude that

$$\int_{0}^{2\pi} d\mu(\theta_{2m}) \frac{1}{|t\varphi_{2m-1} + (r - ze^{i\theta_{2m}})\varphi_{2m}|^s} \leq C_\mu(t, s, \epsilon) \left\| \begin{pmatrix} \varphi_{2m-1} \\ \varphi_{2m} \end{pmatrix} \right\|^{-s} .$$

Case 3b: Assume that $t \frac{\varphi_{2m-1}}{\varphi_{2m}} \leq 2M$. First we use (7.2) once more to deduce that

$$\int_{0}^{2\pi} d\mu(\theta_{2m}) \frac{1}{|t\varphi_{2m-1} + (r - ze^{i\theta_{2m}})\varphi_{2m}|^s} \leq \frac{1}{|e^{i\theta_{2m}} - 1/z(r + t\varphi_{2m-1})|^s} \int_{0}^{2\pi} d\theta_{2m} \frac{1}{|\varphi_{2m-1}|^s} \leq 2^s C_\mu^{(1)}(s) .$$

Under the current assumption we have

$$\left\| \begin{pmatrix} \varphi_{2m-1} \\ \varphi_{2m} \end{pmatrix} \right\|^s \leq \left( \frac{4M}{t} + 2 \right)^s |\varphi_{2m}|^s .$$

Hence, there exists $0 < C_\mu(s, t) < \infty$ such that

$$\int_{0}^{2\pi} d\mu(\theta_{2m}) \frac{1}{|t\varphi_{2m-1} + (r - ze^{i\theta_{2m}})\varphi_{2m}|^s} \leq C_\mu(s, t) \left\| \begin{pmatrix} \varphi_{2m-1} \\ \varphi_{2m} \end{pmatrix} \right\|^{-s} .$$
Therefore, combining all the previous cases, it follows that we always have
\[
\int_0^{2\pi} d\mu(\theta_{2m}) \frac{1}{|t\varphi_{2m-1}^n + (r - ze^{i\theta_{2m}})|^{2n+1}} \leq C_\mu(s, t, \epsilon) \left\| T_z(\omega, m - n) \begin{pmatrix} \varphi_{2n-1}^{2n} \\ \varphi_{2n}^{2n} \end{pmatrix} \right\|^{-s}.
\]

Thus, (7.12) now gives that
\[
E(\|G_z^{[2n,2m]}(2n, 2m)\|^s) \leq C_\mu(s, t, \epsilon) E\left[\left\| T_z(\omega, m - n) \begin{pmatrix} \varphi_{2n-1}^{2n} \\ \varphi_{2n}^{2n} \end{pmatrix} \right\|^{-s}\right].
\]

Since \(\varphi^{2n}\) is a solution of \((U_\omega - z)\psi = 0\) that satisfies the 0-boundary condition at \(2n\) with \(\varphi_{2n}^{2n} = 1\), a straightforward calculation shows that
\[
\varphi_{2n-1}^{2n} = \frac{1 - r}{t}.
\]

Using this along with Lemma A.1, with \(\Lambda = \{x \in \mathbb{C} : |x| - 1| \leq 1/2\}\) and
\[
v = \begin{pmatrix} \varphi_{2n-1}^{2n} \\ \varphi_{2n}^{2n} \end{pmatrix},
\]

it follows that there exist \(\alpha_1 > 0\) and \(s_0 \in (0, 1)\) and \(\bar{C}_\mu^{(1)}(s_0, t, \epsilon) < \infty\) such that
\[
E[\|G_z^{[2n,2m]}(2n, 2m)\|^s] \leq \bar{C}_\mu^{(1)}(s_0, t, \epsilon) e^{-\alpha_1(m-n)},
\]
for all \(z \in \{x \in \mathbb{C} : 0 < |x| - 1| < \epsilon\}, 0 < \epsilon < 1/2\) and \(m, n \in \mathbb{Z}\) such that \(m - n \geq 2\).

**Part II:** In the case \(n \geq m + 2\): Let \(\psi^{2n} (\psi^{2m})\) be the solution of \((U - z)\psi = 0\) that satisfies the 0-right(left) boundary condition at \(2n\) \((2m)\) respectively, with \(\psi_{2n}^{2n} = \psi_{2m}^{2m} = 1\). Using (5.8), (5.1) and (7.13), we obtain that
\[
G_z^{[2m,2n]}(2n, 2m) = \frac{1}{z(t\psi_{2m-1}^{2n} + (1 - r)\psi_{2m}^{2n})}.
\]

Using the same procedure as part I, with \(M = 1 - r\), we see that
\[
\int_0^{2\pi} d\mu(\theta_{2m}) \frac{1}{z(t\psi_{2m-1}^{2n} + (1 - r)\psi_{2m}^{2n})} \leq C_\mu(s, t, \epsilon) \left\| T_z(\omega, m - n) \begin{pmatrix} \psi_{2n-1}^{2n} \\ \psi_{2n}^{2n} \end{pmatrix} \right\|^{-s}.
\]
A similar use of Lemma A.1 allows us to conclude that

\[ \mathbb{E}[|G_z^{[2m,2n]}(2n,2m)|^{s_0}] \leq \tilde{C}_\mu^{(2)}(s_0,t,\epsilon)e^{-\alpha_1(n-m)}, \]

for all \( z \in \{ x \in \mathbb{C} : 0 < |x| - 1 | < \epsilon \}, 0 < \epsilon < 1/2 \) and \( m, n \in \mathbb{Z} \) such that \( n - m \geq 2 \).

This along with the result of part I, concludes the proof for \( k, l \in \{2n,2m\} \) and \( |m-n| \geq 2 \).

\[ \square \]

### 7.4. The Full lattice

Now that we have proved the decay of the reduced case, we are ready to prove the required decay estimate for Green’s function \( G_z^\omega(k,l) \).

**Proof of Theorem 7.1.** For \( |k - l| > 4 \): Clearly there exist \( m, n \in \mathbb{Z} \) such that \( k \in \{2n,2n+1\}, l \in \{2m,2m+1\} \) and \( |m-n| > 1 \). Thus using Lemma 7.1 we obtain that there exist \( 0 < \kappa(r,s,\mu) < \infty \) such that

\[
(\mathbb{E}[|G_z^\omega(k,l)|^s])^2 \leq \kappa(r,s,\mu) \sum_{i,j=0}^1 (\mathbb{E}[|G_z^\omega(2n+2i,2m-2j)|^4s])^{1/2}
\]

\[
\leq \kappa(r,s,\mu) \sum_{i,j=0}^1 (\mathbb{E}[|G_z^{[2n+2i,2m-2j]}(2n+2i,2m-2j)|^4s])^{1/2},
\]

where the second inequality follows from Lemma (7.10). Next the result of Lemma (7.3) gives that there exist \( s_0 \in (0,1), 0 < \tilde{C}_1 < \infty, \alpha > 0 \) such that

\[
(\mathbb{E}[|G_z^\omega(k,l)|^s_0])^2 \leq \tilde{C}_1^{(1)}(r,s,\mu) \sum_{i,j=0}^1 e^{-\alpha|2n-2m+2i+2j|}.
\]

Finally noting that by the triangle inequality and the definition of \( m, n \) we have

\[
|2n - 2m + 2i + 2j| \geq \left| |k - l| - |(2n - k) - (2m - l) + 2i + 2j| \right|
\]

\[
\geq |k - l| - 5.
\]

Thus, taking \( \tilde{C}_1 = 4\tilde{C}_1^{(1)}(r,s,\mu) e^{5\alpha} \), it follows that

\[
\mathbb{E}[|G_z^\omega(k,l)|^{s_0}] \leq \tilde{C}_1 e^{-\alpha|k-l|}.
\]
For \( |k - l| \leq 4 \), we use Theorem 6.1 to show that there exists \( 0 < \tilde{C}_2 < \infty \) such that
\[
E[|G_z^{\omega}(k, l)|^s] \leq \tilde{C}_2 e^{4\alpha} e^{-\alpha |k-l|}.
\]
Letting \( C = \max\{\tilde{C}_1, \tilde{C}_2 e^{4\alpha}\} \) gives the required result. \( \square \)

### 7.5. The Half Lattice

As we remarked following Theorem 7.1, a similar estimate holds for \( G^{[a,b]}_z \) for arbitrary \( -\infty \leq a < b \leq \infty \). Since the half lattice case, \( a = 0 \) and \( b = \infty \), is closely related to Orthogonal polynomials on the Unit circle, see section 9.1, in what follows we prove the fractional moment condition for this specific case, i.e. we show that:

**Theorem 7.2.** Assume that \( \{\theta_\omega^k\}_{k \in [0, \infty)} \) are i.i.d. with probability measure \( d\mu(\theta) = \tau(\theta) d\theta \), where \( \tau \in L_\infty(\mathbb{T}) \) and \( 0 < \epsilon < 1/2 \). There exist \( s \in (0, 1/8) \), \( 0 < C < \infty \) and \( \alpha > 0 \) such that
\[
(7.14) \quad E[|G_z^{[0, \infty)}(k, l)|^s] \leq Ce^{-\alpha |k-l|},
\]
for all \( z \in \{x \in \mathbb{C} : 0 < |x| - 1 | < \epsilon \} \), and \( k, l \in [0, \infty) \).

**Proof.** **Part I:** For \( k, l \in \mathbb{N} \setminus \{1\} \), the proof follows exactly as in the full lattice case.

**Part II:** Next we deal with the boundary terms; namely when either \( k \) or \( l \) belongs to \( \{0, 1\} \).

In case \( k = 1 \), we use Theorem 5.2 along with (2.22), to obtain
\[
G_{[0, \infty)}^{[0, \infty)}(1, l) = \left( \frac{r^2}{t^2} (1 + e^{-i(\theta_\omega^2 - \theta_\omega^3)} - \frac{e^{-\theta_\omega^2}}{z}) - \frac{ze^{i\theta_\omega^2}}{t^2} \right) G_{[0, \infty)}^{[0, \infty)}(3, l) - \frac{r}{t} (1 - \frac{e^{-i\theta_\omega^2}}{z}) G_{[0, \infty)}^{[0, \infty)}(4, l).
\]
Therefore, it follows that that for all \( l \)
\[
E[|G_{[0, \infty)}^{[0, \infty)}(1, l)|^s] \leq c_1(r, s, \epsilon) \left( E[|G_{[0, \infty)}^{[0, \infty)}(3, l)|^s] + E[|G_{[0, \infty)}^{[0, \infty)}(4, l)|^s] \right) \]
\[
\leq C_1(e^{2\alpha} + e^{3\alpha}) e^{-\alpha (l-1)},
\]
where the last inequality follows from part I and Theorem 6.1.
For $k = 0$, Theorem 5.2 combined with (2.23) gives

$$E[G^{[0,\infty)}(0, l)|s] = E\left[\frac{t}{re^{i\theta}} - \frac{1}{z}\right]^s|G^{[0,\infty)}(1, l)|^s \leq C_2e^{-\alpha l},$$

which follows from an application of Hölder’s inequality, (7.2) and the previous case.

The cases when $l \in \{0, 1\}$ are treated similarly. □

**Remark 7.5.1.** The method presented above can be easily extended to arbitrary $-\infty \leq a, b \leq \infty$. The only difference arises in dealing with the boundary terms, i.e. $k, l \in \{a, a + 1, b - 1, b\}$, where (2.23) must be replaced with the appropriate boundary relations.
CHAPTER 8

THE ROAD TO DYNAMICAL LOCALIZATION

In this chapter we show how exponential decay of fractional moments of Green’s function, (7.1), implies strong dynamical localization for the type of random unitary model we are studying. The method introduced in this chapter is by no means restricted to dimension one, therefore we will state our results for arbitrary dimension $d$.

First we introduce the $d$ dimensional generalization of the operator $U_{\omega} (2.5)$. Following [27], we view $l^2(\mathbb{Z}^d)$ as $\otimes_{j=1}^{d} l^2(\mathbb{Z})$ so that for all $k \in \mathbb{Z}^d$, $e_k \simeq e_{k_1} \otimes ... \otimes e_{k_d}$. As before, we introduce the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega$ is now identified with $\mathbb{T}^{\mathbb{Z}^d}$, $\mathcal{F}$ is the $\sigma$-algebra generated by cylinders of Borel sets, and $\mathbb{P} = \otimes_{k \in \mathbb{Z}^d} \mu$, where $\mu$ is a non trivial probability measure on $\mathbb{T}$.

The random variables $\theta_k$ on $(\Omega, \mathcal{F}, \mathbb{P})$ are defined by

\begin{equation}
\theta_k : \Omega \to \mathbb{T}, \quad \theta_k^\omega = \omega_k, \quad k \in \mathbb{Z}^d.
\end{equation}

The diagonal operator $D_{\omega}(d)$ is given by

\begin{equation}
D_{\omega}(d)e_k = e^{-i\theta_k^\omega}e_k,
\end{equation}

while $S_d(t)$ is defined using $S$ from (2.2) by

\begin{equation}
S_d(t) = \otimes_{j=1}^{d} S(t_j),
\end{equation}

characterized by $t = (t_1, ..., t_d) \in (0,1)^d$. The unitary operator $U_{\omega}(d)$ on $l^2(\mathbb{Z}^d)$ is then defined as

\begin{equation}
U_{\omega}(d) = D_{\omega}(d)S_d(t),
\end{equation}
Using this definition of $S_d(t)$ it is easy to see that it has a purely absolutely continuous spectrum given by

$$\sigma(S_d(t)) = \sigma(S(t_1)) \times \sigma(S(t_2)) \times \ldots \times \sigma(S(t_d)).$$

Using the maximum norm on $\mathbb{Z}^d$, we see that $S_d(t)$ inherits the band structure of $S$ such that

$$\langle e_k, S_d(t)e_l \rangle = 0 \quad \text{if } |k - l| > 2.$$ 

More discussion on the properties of this $d$-dimensional generalization of the model can be found in [27].

8.1. A Second Moment Estimate

Now in order to prove strong dynamical localization, we start by showing how exponential decay of the expectation of fractional moments of matrix elements of the resolvent implies a decay estimate of the expectation of their second moment. A similar result was proven for the self-adjoint Anderson model in [22].

**Theorem 8.1.** Assume that $\{\theta_k^x\}_{k \in \mathbb{Z}^d}$ are i.i.d. with probability measure $d\mu(\theta) = \tau(\theta)d\theta$, where $\tau \in L^\infty(\mathbb{T})$ and $0 < \epsilon < 1/2$. Furthermore, assume that there exist $s \in (0, 1)$, $0 < C < \infty$ and $\alpha > 0$ such that

$$E[|\langle e_k, (U_\omega(d) - z)^{-1}e_l \rangle|^s] \leq Ce^{-\alpha|k-l|},$$

for all $z \in \{x \in \mathbb{C} : 1 - \epsilon < |x| < 1\}$, and $k, l \in \mathbb{Z}^d$. Then there exists $0 < \tilde{C} < \infty$ such that following condition holds

$$E[(1 - |z|^2)|\langle e_k, (U_\omega(d) - z)^{-1}e_l \rangle|^2] \leq \tilde{C}e^{-\alpha|k-l|},$$

for all $z \in \{x \in \mathbb{C} : 1 - \epsilon < |x| < 1\}$ and $k, l \in \mathbb{Z}^d$.

**Proof.** Since the following argument is independent of the dimension $d$ we will drop the $d$ dependence in order to simplify the notation.
For \( \delta \in \mathbb{T} \), let \( \eta_k = e^{-i(\theta^* \omega + \delta)} - e^{-i\theta^* \omega} \). Then define \( D^\delta_\omega = D_\omega + \eta_k P_k \), where \( P_k \) is the orthogonal projection into the span of \( e_k \). Let \( U^\delta_\omega = D^\delta_\omega S \). Using the resolvent identity we have

\[
(U^\delta_\omega - z)^{-1} - (U_\omega - z)^{-1} = -(U^\delta_\omega - z)^{-1} \eta_k P_k (U_\omega - z)^{-1},
\]

for all \( z \in \mathbb{C} \) such that \( 0 < |z| < 1 \). Letting \( F(z) = S(U_\omega - z)^{-1} \) and \( F^\delta(z) = S(U^\delta_\omega - z)^{-1} \), the last equation takes the form

\[
F^\delta(z) - F(z) = -F(z) \eta_k P_k F(z).
\]

Denoting \( F(i, j, z) = \langle e_i, F(z)e_j \rangle \) and using that \( \langle x, cy \rangle = c \langle x, y \rangle \), it is easy to see that

\[
F^\delta(z) - F(z) = -\eta_k \frac{F(z) P_k F(z)}{1 + \eta_k F(k, k, z)}.
\]

Therefore, for all \( l \in \mathbb{Z}^d \)

\[
F^\delta(k, l, z) = \frac{F(k, l, z)}{1 + \eta_k F(k, k, z)}.
\]

On the other hand, we also have that

\[
|F^\delta(k, l, z)|^2 \leq \sum_{y \in \mathbb{Z}} |F^\delta(k, y, z)|^2
\]

\[
= \langle e_k, S(U^\delta_\omega - z)^{-1}[(U^\delta_\omega - z)^{-1}]^* S^* e_k \rangle.
\]

Since \( U^\delta_\omega \) is a unitary operator, the following identity holds

\[
[(U^\delta_\omega - z)^{-1}]^* = -\frac{1}{z} (U^\delta_\omega - \frac{1}{z})^{-1} U^\delta_\omega.
\]

Thus, it follows that

\[
|F^\delta(k, l, z)|^2 \leq -\frac{1}{z} \langle e_k, S(U^\delta_\omega - z)^{-1}(U^\delta_\omega - \frac{1}{z})^{-1} D^\delta_\omega e_k \rangle
\]

\[
= -e^{-i(\theta^* \omega + \delta)} \langle e_k, S(U^\delta_\omega - z)^{-1}(U^\delta_\omega - \frac{1}{z})^{-1} e_k \rangle.
\]

Again using the resolvent identity, we see that

\[
(U^\delta_\omega - z)^{-1}(U^\delta_\omega - \frac{1}{z})^{-1} = \frac{z}{|z|^2 - 1} \{(U^\delta_\omega - z)^{-1} - (U^\delta_\omega - \frac{1}{z})^{-1}\}.
\]
Hence,

$$\left| F_\delta(k, l, z) \right|^2 \leq \frac{e^{-i(\theta_k^\delta + \delta)}}{1 - |z|^2} \left\{ \langle e_k, S(U_\omega^\delta - z)^{-1} e_k \rangle - \langle e_k, S(U_\omega^\delta - \frac{1}{z})^{-1} e_k \rangle \right\}.$$ 

From (8.8), the definition of $U_\omega^\delta$ and the fact that $(U_\omega^\delta - z)^{-1} = -\frac{1}{z}[I - U_\omega^\delta(U_\omega^\delta - z)^{-1}]$, it follows that

$$\langle e_k, S(U_\omega^\delta - 1)^{-1} e_k \rangle = -\bar{z}e^{i(\theta_k^\delta + \delta)}\left\{ 1 - \langle U_\delta^\omega(U_\omega^\delta - 1)^{-1} e_k, e_k \rangle \right\} = e^{i(\theta_k^\delta + \delta)} \left\{ 1 - e^{i(\theta_k^\delta + \delta)} F_\delta(k, k, z) \right\}.$$ 

Therefore, we now obtain that

$$\left| F_\delta(k, l, z) \right|^2 \leq \frac{1}{1 - |z|^2} \left\{ 2\Re e^{-i(\theta_k^\delta + \delta)} F_\delta(k, k, z) - 1 \right\} = \frac{1}{1 - |z|^2} \left\{ \left| F_\delta(k, k, z) \right|^2 - \left| e^{i(\theta_k^\delta + \delta)} - F_\delta(k, k, z) \right|^2 \right\},$$

since $|x - y|^2 = |x|^2 + |y|^2 - 2\Re xy$. Using (8.7), to rewrite $F_\delta(k, k, z)$ in terms of elements of $F$, along with the definition of $\eta_k$ we get

$$\left| F_\delta(k, l, z) \right|^2 \leq \frac{1}{1 - |z|^2} \left\{ \left| F(k, k, z) \right|^2 - \left| e^{i\theta_k^\delta} - F(k, k, z) \right|^2 \right\}. \left\{ 1 + \eta_k F(k, k, z) \right\}.$$

This inequality gives, in particular, that $F(k, k, z) \neq 0$. Therefore

$$(8.9) \quad (1 - |z|^2) \left| F_\delta(k, l, z) \right|^2 \leq \frac{1}{1 - e^{i\theta_k} F(k, k, z) - 1}.$$ 

Finally note that writing the inequality in this form allows us to conclude that $|1 - e^{i\theta_k} F(k, k, z) - 1| \leq 1$.

One can also use the fact that

$$\left| F_\delta(k, k, z) \right| \leq \left\| (U_\omega^\delta - z)^{-1} \right\| \leq \frac{1}{1 - |z|},$$

for all $\delta \in \mathbb{T}$, to get a different upper bound on $(1 - |z|^2)\left| F_\delta(k, l, z) \right|^2$. Since (8.7) can be rewritten as

$$(8.10) \quad F_\delta(k, l, z) = \frac{1}{\eta_k + F(k, k, z)^{-1}} F(k, k, z),$$
it follows that \(1 - |\eta_k + F(k, k, z)| \leq |z|\). Then by choosing \(\delta\) such that \(e^{-i\delta} = \frac{1 - e^{i\theta_k} F(k, k, z)^{-1}}{|1 - e^{i\theta_k} F(k, k, z)|}\), we see that
\[
|1 - e^{i\theta_k} F(k, k, z)^{-1}| \leq |z|.
\]

Using this along with (8.10) we obtain the following upper bound
\[
(8.11) \quad (1 - |z|^2)|F_\delta(k, l, z)|^2 \leq \frac{1 - |1 - e^{i\theta_k} F(k, k, z)^{-1}|^2}{|\eta_k + F(k, k, z)|^2} |F(k, l, z)|^2 |F(k, k, z)|^2.
\]

Combining the two estimates (8.9) and (8.11) and using that for \(0 < s < 1\) we have \(\min(1, |x|^2) \leq |x|^s\), it follows that
\[
(1 - |z|^2)|F_\delta(k, l, z)|^2 \leq \frac{1 - |1 - e^{i\theta_k} F(k, k, z)^{-1}|^2}{|e^{-i\delta} - (1 - e^{i\theta_k} F(k, k, z))|^2} |F(k, l, z)|^s |F(k, k, z)|^s.
\]

Letting \(y = 1 - e^{i\theta_k} F(k, k, z)^{-1}\), this can be rewritten as
\[
(1 - |z|^2)|F_\delta(k, l, z)|^2 \leq \frac{(1 - |y|^2)|1 - y|^s}{|e^{-i\delta} - y|^2} |F(k, l, z)|^s.
\]

Since the expectations of \(F\) and \(F_\delta\) are related by
\[
\mathbb{E}[|F(k, l, z)|^2] = \mathbb{E}\left[ \int d\mu(\theta_k + \delta)|F_\delta(k, l, z)|^2 \right],
\]
it follows that
\[
\mathbb{E}[(1 - |z|^2)|F(k, l, z)|^2] \leq ||\tau||_{\infty} \mathbb{E}\left[ |F(k, l, z)|^s \sup_{\{y \in \mathbb{C}: |y| < 1\}} \int_0^{2\pi} d\delta \frac{(1 - |y|^2)|1 - y|^s}{|e^{-i\delta} - y|^2} \right]
\leq 2^s ||\tau||_{\infty} \mathbb{E}\left[ |F(k, l, z)|^s \sup_{\{y \in \mathbb{C}: |y| < 1\}} \int_0^{2\pi} d\delta \frac{(1 - |y|^2)}{|e^{-i\delta} - y|^2} \right].
\]

Next we evaluate the integral
\[
\int_0^{2\pi} d\delta \frac{(1 - |y|^2)}{|e^{-i\delta} - y|^2} = \int_0^{2\pi} d\delta \Re\left[\frac{e^{-i\delta} + y}{e^{-i\delta} - y}\right] = \Re\left[\int_0^{2\pi} d\delta \left[\frac{2e^{-i\delta}}{e^{-i\delta} - y} - 1\right]\right].
\]

The latter integral can be easily evaluated using Cauchy integral formula, by simply substituting \(z = e^{-i\delta}\) and gives
\[
\int_0^{2\pi} d\delta \frac{(1 - |y|^2)}{|e^{-i\delta} - y|^2} = 2\pi.
\]
Therefore,

\[(8.12) \quad \mathbb{E}[(1 - |z|^2)|F(k, l, z)|^2] \leq 2^{s+1} \pi \|\tau\|_\infty \mathbb{E}[|F(k, l, z)|^s].\]

Since \(S_d(t)\) is a unitary operator with band structure, we see that for \(s \in (0, 1)\) such that condition (8.5) holds

\[
\mathbb{E}[|F(k, l, z)|^s] = \mathbb{E}[\langle S_d(t)^* e_k, (U_\omega(d) - z)^{-1} e_l \rangle^s] \\
\leq C_1(r, s) \sum_{|m-k| \leq 2} \mathbb{E}[\langle e_m, (U_\omega(d) - z)^{-1} e_l \rangle^s] \\
\leq C_2(r, s)e^{-\alpha|k-l|}.
\]

Combining this with (8.12), we obtain

\[
\mathbb{E}[(1 - |z|^2)|F(k, l, z)|^2] \leq \tilde{C}_1 e^{-\alpha|k-l|}.
\]

Finally, in order to get the required decay of the second moment of elements of \((U_\omega(d) - z)^{-1}\) we use again that \(S_d(t)\) is a unitary operator with band structure. Therefore, for \(k, l \in \mathbb{Z}^d\)

\[
\mathbb{E}[(1 - |z|^2)\langle e_k, (U_\omega(d) - z)^{-1} e_l \rangle^2] = \mathbb{E}[(1 - |z|^2)\langle Se_k, S(U_\omega(d) - z)^{-1} e_l \rangle^2] \\
\leq C_3(r, s) \sum_{|m-k| \leq 2} \mathbb{E}[(1 - |z|^2)|F(m, l, z)|^2] \\
\leq \tilde{C} e^{-\alpha|k-l|},
\]

where \(\tilde{C} = C_3 \tilde{C}_1\), which gives the required result. \(\Box\)

### 8.2. Strong Dynamical Localization for Absolutely Continuous Distributions

First we show that the above decay estimate implies strong dynamical localization, namely

**Theorem 8.2.** Consider \(U_\omega(d) = D_\omega(d)S_d(t)\), for which the hypothesis of Theorem 8.1 holds, then there exist \(0 < \tilde{C}_1 < \infty\) and \(\beta > 0\) such that

\[(8.13) \quad \mathbb{E}\left[\sup_{n \in \mathbb{Z}} |\langle e_k, [U_\omega(d)]^n e_l \rangle|\right] \leq \tilde{C} e^{-\beta|k-l|},\]
for all $k, l \in \mathbb{Z}^d$.

Before proving the Theorem we prove the following Lemma that plays a central role in the proof.

**Lemma 8.1.** Let $U$ be a unitary operator. Then for $n \in \mathbb{Z}$,

$$U^n = \lim_{r \rightarrow 1^+} \frac{1 - r^2}{2\pi} \int_0^{2\pi} (U - re^{i\theta})^{-1}(U^{-1} - re^{-i\theta})^{-1} e^{i\theta n} d\theta. \tag{8.14}$$

**Proof.** Using that $U$ is unitary, we have

$$(U^{-1} - re^{-i\theta})^{-1} = -\frac{1}{r} e^{i\theta} U (U - \frac{1}{r} e^{i\theta})^{-1}.\]$$

Therefore, using the resolvent identity, we obtain

$$(1 - r^2)(U - re^{i\theta})^{-1}(U^{-1} - re^{-i\theta})^{-1} = [r(U - re^{i\theta})^{-1} - \frac{1}{r} (U - \frac{1}{r} e^{i\theta})^{-1}] e^{i\theta}.$$\]

Thus we have

$$\frac{1 - r^2}{2\pi} \int_0^{2\pi} (U - re^{i\theta})^{-1}(U^{-1} - re^{-i\theta})^{-1} e^{i\theta n} d\theta = \int_0^{2\pi} [r(U - re^{i\theta})^{-1} - \frac{1}{r} (U - \frac{1}{r} e^{i\theta})^{-1}] e^{i\theta (n+1)} \frac{d\theta}{2\pi}.$$\]

Next, we evaluate the integrals on the right hand side. Since,

$$\int_0^{2\pi} r(U - re^{i\theta})^{-1} e^{i\theta (n+1)} \frac{d\theta}{2\pi} = \int_0^{2\pi} U^{-1}(r^{-1} - e^{i\theta} U^{-1})^{-1} e^{i\theta (n+1)} \frac{d\theta}{2\pi}$$\]

$$= \sum_{k=0}^{\infty} r^{k+1} U^{-k-1} \int_0^{2\pi} e^{i\theta (k+n+1)} \frac{d\theta}{2\pi}$$\]

$$= \sum_{k=0}^{\infty} r^{k+1} U^{-k-1} \delta_{k,-(n+1)},$$

it follows that

$$\int_0^{2\pi} r(U - re^{i\theta})^{-1} e^{i\theta (n+1)} \frac{d\theta}{2\pi} = \begin{cases} \frac{r^{-n} U^n}{2\pi} & \text{if } n < 0 \\ 0 & \text{if } n \geq 0. \end{cases}$$
Similarly, we have that
\[-\int_0^{2\pi} \frac{1}{r} (U - 1) e^{i\theta} - 1 e^{i\theta(n+1)} d\theta = \sum_{k=0}^{\infty} r^k U^k \int_0^{2\pi} e^{-i\theta(n-k)} d\theta = \sum_{k=0}^{\infty} r^k U^k \delta_{k,n} .\]

Thus, the integral gives
\[-\int_0^{2\pi} \frac{1}{r} (U - 1) e^{i\theta} - 1 e^{i\theta(n+1)} d\theta = \begin{cases} r^n U^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0. \end{cases}\]

Combining the two integrals and taking the limit as \( r \to 1^+ \) gives the required result. \( \square \)

**Remark 8.2.1.** The previous is an elementary proof of a more general result, namely that for all \( f \in C(\mathbb{T}) \) and a unitary operator \( U \)

\[ f(U) = \lim_{r \to 1^+} \frac{1 - r^2}{2\pi} \int_0^{2\pi} (U - re^{i\theta})^{-1}(U^{-1} - re^{-i\theta})^{-1} f(e^{i\theta}) d\theta. \]

This result can be proven using the spectral theorem and Poisson integral.

Now we are ready to show that under the assumptions of Theorem 8.1, i.e. exponential decay of fractional moments, the model is strongly dynamically localized.

**Proof of Theorem 8.2.** Since \( |\langle e_k, (U - re^{i\theta})^{-1}(U^{-1} - re^{-i\theta})^{-1} e_l \rangle| \leq \frac{1}{(1 - r)^2} \), we can use Fubini’s Theorem along with the previous Lemma and continuity of the inner product to conclude that for all \( k, l \in \mathbb{Z}^d \) and all \( n \in \mathbb{Z} \)

\[ \langle e_k, U^n e_l \rangle = \lim_{r \to 1^+} (1 - r^2) \int_0^{2\pi} e^{i\theta n} \langle e_k, (U - re^{i\theta})^{-1}(U^{-1} - re^{-i\theta})^{-1} e_l \rangle d\theta = \lim_{r \to 1^+} (1 - r^2) \int_0^{2\pi} e^{i\theta n} \sum_{j \in \mathbb{Z}^d} \langle (U - re^{i\theta})^{-1} e_k, e_j \rangle \langle e_j, (U - re^{-i\theta})^{-1} e_l \rangle, \]

where we used that \( [(U^{-1} - re^{-i\theta})^{-1}]^* = (U - re^{i\theta})^{-1} \).

Therefore, using Fubini’s Theorem and Fatou’s Lemma, we have

(8.15)

\[ \mathbb{E} \sup_{n \in \mathbb{Z}} |\langle e_k, U^n e_l \rangle| \leq \liminf_{r \to 1^+} \int_0^{2\pi} \frac{d\theta}{2\pi} \sum_{j \in \mathbb{Z}^d} \mathbb{E} \{ |1 - r^2| \langle e_j, (U - re^{i\theta})^{-1} e_k \rangle |\langle e_j, (U - re^{-i\theta})^{-1} e_l \rangle| \}. \]
Using Hölder’s inequality, we see that
\[
\mathbb{E}\{(1 - r^2)|\langle e_j, (U_\omega - re^{i\theta})^{-1}e_k\rangle||\langle e_j, (U_\omega - re^{i\theta})^{-1}e_l\rangle|\}
\leq (\mathbb{E}\{(1 - r^2)|\langle e_j, (U_\omega - re^{i\theta})^{-1}e_k\rangle|^2\})^{1/2}
\times (\mathbb{E}\{(1 - r^2)|\langle e_j, (U_\omega - re^{i\theta})^{-1}e_l\rangle|^2\})^{1/2}
\leq Ce^{-\beta|k-j|+|l-j|},
\]
where the last inequality, with \(0 < C < \infty\), follows from Theorem 8.1.

Inserting this estimate in (8.15) and using the triangle inequality, we obtain
\[
\mathbb{E}[\sup_{n \in \mathbb{Z}} |\langle e_k, U^n_\omega e_l\rangle|] \leq Ce^{-\beta/2|k-l|} \sum_{j \in \mathbb{Z}^d} e^{-\beta/2|k-j|+|l-j|}.
\]
Finally, using Schwarz inequality we see that
\[
\sum_{j \in \mathbb{Z}^d} e^{-\beta/2|k-j|+|l-j|} \leq C_1(d, \beta),
\]
which ends the proof. \(\square\)

**Remark 8.2.2.** The \(d\)-dimensional model can also be defined on \(l^2([a,b]^d)\) for \(-\infty \leq a, b \leq \infty\) using the same procedure as before with \(S_{[a,b]}^{[\eta_a, \eta_b]}\) replacing \(S\) and \(D_{[a,b]}(d)\) defined appropriately. It is straightforward to see that the results presented in this chapter remain true for \(U_{[a,b]}^{[\eta_a, \eta_b]}(d)\).

Applying the previous Theorem, we obtain dynamical localization of the unitary Anderson model with absolutely continuous distribution of random phases in two regimes, namely

**Theorem 8.3.** Let \(U_\omega(d)\) be defined by (8.4). Assume that \(\{\theta^\omega_k\}_{k \in \mathbb{Z}^d}\) are i.i.d. with probability distribution \(d\mu(\theta) = \tau(\theta)d\theta\), where \(\tau \in L^\infty(\mathbb{T})\). Then the following is true:

(i) For \(d = 1\), the dynamical localization property (8.13) holds for all \(t \in (0, 1)\) for both \(U_\omega\) and \(U_\omega^{[0, \infty)}\).

(ii) For \(d > 1\), there exists \(t_0 > 0\) such that if \(|t| < t_0\), property (8.13) holds.
Proof. (i) This follows directly from Theorems 8.2 combined with Theorem 7.1 and Theorem 7.2 respectively.

(ii) This again follows from Theorem 8.2 and the fractional moment estimate proven in Theorem 2.1 of [27].

We conclude this chapter by proving a consequence of Theorem 8.2 concerning moments of the position operator $X$ in energy space, defined by $(X\psi)(x) = x\psi(x)$.

Corollary 8.1. For $U_\omega(d)$, for which the hypothesis of Theorem 8.1 holds, and for all $p > 0$ and $\psi_n = [U_\omega(d)]^n|e_0\rangle$, the $p$-moment of the position operator $X$ is bounded for all time, i.e.

$$\mathbb{E}\{\sup_{n \in \mathbb{Z}} \langle |X|^p \psi_n, \psi_n \rangle \} < \infty.$$ 

Proof. Since $U_\omega^n$ is a unitary operator, we have $|\langle e_k, U_\omega^n e_l \rangle|^2 \leq |\langle e_k, U_\omega e_l \rangle|$. Thus,

$$\sum_{x \in \mathbb{Z}^d} |x|^p |\psi_n(x)|^2 \leq \sum_{x \in \mathbb{Z}^d} |x|^p |\langle e_x, U_\omega^n e_0 \rangle|.$$ 

Therefore, using Fubini’s Theorem, we have

$$\mathbb{E}\{\sup_n \langle |X|^p \psi_n, \psi_n \rangle \} \leq \mathbb{E}\{\sum_{x \in \mathbb{Z}^d} |x|^p \sup_{n \in \mathbb{Z}} |\langle e_x, U_\omega^n e_0 \rangle|\} \leq \sum_{x \in \mathbb{Z}^d} |x|^p \mathbb{E}\{\sup_{n \in \mathbb{Z}} |\langle e_x, U_\omega^n e_0 \rangle|\}.$$ 

Thus, it follows by Theorem 8.2 that

$$\mathbb{E}\{\sup_n \langle |X|^p \psi_n, \psi_n \rangle \} \leq \tilde{C} \sum_{x \in \mathbb{Z}^d} |x|^p e^{-\beta|x|} < \infty,$$

which concludes the proof. 

□
CHAPTER 9

CONCLUDING REMARKS

9.1. Application to OPUC

The extension of our result to $l^2(N_0)$, with $N_0 = \mathbb{N} \cup \{0\}$, was intended to pave the way for the applications of our localization results to orthogonal polynomials on the unit circle (OPUC) with respect to an infinitely supported probability measure $d\mu$. Such polynomials $\Phi_k$ are determined via the Szego recursion $\Phi_{k+1}(z) = z\Phi_k(z) - \sigma_k \Phi^*_k(z)$, $\Phi_0 = 1$, by a sequence of complex valued coefficients $\{\alpha_k\}_{k \in \mathbb{N}_0}$, such that $|\alpha_k| < 1$, called Verblunsky coefficients, which also characterize the measure $d\mu$, see [35, 36]. This latter relation is encoded in a five diagonal unitary matrix $C$ on $l^2(\mathbb{N}_0)$ representing multiplication by $z \in S^1$: The measure $d\mu$ arises as the spectral measure $\mu(\Delta) = \langle e_0, E(\Delta)e_0 \rangle$ of the cyclic vector $e_0$ of $C$. This matrix is the equivalent of the Jacobi matrix in the case of orthogonal polynomials with respect to a measure on the real axis, and it is called the CMV matrix, after [10].

In case the Verblunsky coefficients all have the same modulus, i.e.

$$\alpha_k = re^{i\eta_k}, \quad k = 0, 1, \ldots$$

the corresponding CMV matrix reads

$$C = \begin{pmatrix}
  r e^{-i\eta_0} & r t e^{-i\eta_1} & t^2 \\
  t & -r^2 e^{i(\eta_0 - \eta_1)} & -r t e^{i\eta_0} \\
  r t e^{-i\eta_2} & -r^2 e^{i(\eta_1 - \eta_2)} & r t e^{-i\eta_3} \\
  t^2 & -r t e^{i\eta_1} & -r^2 e^{i(\eta_2 - \eta_3)} & -r t e^{i\eta_2} \\
  r t e^{-i\eta_4} & -r^2 e^{i(\eta_3 - \eta_4)} & r t e^{i\eta_3} \\
  t^2 & -r t e^{i\eta_2} & -r^2 e^{i(\eta_4 - \eta_3)} & -r t e^{i\eta_3} \\
  \vdots & & & \ddots
\end{pmatrix}.$$
Now, changing from the canonical basis \( \{ e_j \}_{j \in \mathbb{N}_0} \) to \( \{ e^{i\beta_j} e_j \}_{j \in \mathbb{N}_0} \) by means of the unitary \( B \) defined by \( Be_j = e^{i\beta_j} e_j \), \( j = 0, 1, \ldots \), we get

\[
B^{-1}CB = \begin{pmatrix}
re^{-i\theta_0} & re^{-i\theta_0} & t^2 e^{-i\theta_0} \\
re^{-i\theta_1} & -re^{-i\theta_0} t & t^2 e^{-i\theta_0} \\
-rc^{i(\beta_0 - \beta_1)} & -r^2 e^{i(\eta_0 - \eta_1)} & -rte^{-i\theta_0} e^{i(\beta_2 - \beta_1)} \\
t e^{i(\beta_0 - \beta_1)} & -r^2 e^{i(\eta_0 - \eta_1)} & -r^2 e^{i(\eta_0 - \eta_2)} \\
t e^{i(\beta_0 - \beta_1)} & -rte^{-i\theta_0} e^{i(\beta_2 - \beta_1)} & -rte^{-i\theta_0} e^{i(\beta_3 - \beta_2)} \\
t^2 e^{i(\beta_0 - \beta_1)} & -rte^{-i\theta_0} e^{i(\beta_3 - \beta_2)} & -rte^{-i\theta_0} e^{i(\beta_4 - \beta_3)} \\
\end{pmatrix}.
\]

Then, by choosing the \( \beta_j \)'s suitably, the matrix (9.3) becomes the negative of a matrix of the form of \( U^{(0, \infty)}_\omega \) (2.21):

\[
-U^{(0, \infty)} = \begin{pmatrix}
re^{-i\theta_0} & -re^{-i\theta_0} t & t^2 e^{-i\theta_0} \\
re^{-i\theta_1} & -re^{-i\theta_0} t & t^2 e^{-i\theta_0} \\
-rc^{i(\beta_0 - \beta_1)} & -r^2 e^{i(\eta_0 - \eta_1)} & -rte^{-i\theta_0} e^{i(\beta_2 - \beta_1)} \\
t e^{i(\beta_0 - \beta_1)} & -r^2 e^{i(\eta_0 - \eta_1)} & -r^2 e^{i(\eta_0 - \eta_2)} \\
t e^{i(\beta_0 - \beta_1)} & -rte^{-i\theta_0} e^{i(\beta_3 - \beta_2)} & -rte^{-i\theta_0} e^{i(\beta_4 - \beta_3)} \\
t^2 e^{-i\theta_0} & -rte^{-i\theta_0} e^{i(\beta_3 - \beta_2)} & -rte^{-i\theta_0} e^{i(\beta_5 - \beta_4)} \\
\end{pmatrix}.
\]

Here, as seen from the diagonal elements, the phases \( \theta_k \) are given in terms of the phases of the Verblunsky coefficients by (set \( \eta_{-1} = 0 \))

\[
(9.5) \quad \theta_k = \eta_k - \eta_{k-1}, \quad k = 0, 1, 2, \ldots,
\]

or, equivalently,

\[
(9.6) \quad \eta_k = \theta_k + \theta_{k-1} + \cdots + \theta_0, \quad k = 0, 1, 2, \ldots
\]

The terms in \( t^2 \) require

\[
\beta_1 - \beta_0 = \theta_1 + \pi
\]

\[
\beta_{2k+1} - \beta_{2k-1} = \theta_{2k+1}, \quad k = 1, 2, \ldots
\]

\[
(9.7) \quad \beta_{2k+2} - \beta_{2k} = -\theta_{2k}, \quad k = 0, 1, \ldots
\]
where $\beta_0$ is free. Explicitly, for $k \geq 0$,

$$
\beta_{2k+1} = \theta_{2k+1} + \theta_{2k-1} + \cdots + \theta_1 + \beta_0 + \pi
$$

(9.8)

$$
\beta_{2k+2} = -(\theta_{2k} + \theta_{2k-2} + \cdots + \theta_0).
$$

It is straightforward to check that (9.5) and (9.7) form a consistent choice in the sense that all terms in $rt$ in (9.3) and (9.4) agree. Assuming the $\theta_k$’s are i.i.d. random variables, Theorems 4.2 and 8.3 apply to this case and yield

**Proposition 9.1.** Let $\alpha_k(\omega)_{k \in \mathbb{N}_0}$ be random Verblunsky coefficients of the form

$$
\alpha_k(\omega) = re^{i\eta_k(\omega)}, \quad 0 < r < 1, \quad k = 0, 1, 2, \ldots
$$

(9.9)

whose phases are distributed on $\mathbb{T}$ according to

$$
\eta_k(\omega) \sim d\mu \ast d\mu \ast \cdots \ast d\mu, \quad (k + 1 \text{ convolutions})
$$

(9.10)

and $d\mu$ is a probability measure with non-trivial a.c. component. Then, the random measure $d\mu_\omega$ on $S^1$ with respect to which the corresponding random polynomials $\Phi_{k,\omega}$ are orthogonal is almost surely pure point.

Moreover, if $d\mu(\theta) = \tau(\theta)d\theta$, where $\tau \in L^\infty(\mathbb{T})$, then the corresponding CMV matrix is dynamically localized.

**Remark 9.1.1.** Other localization results for random polynomials on the unit circle, [37], [41], [40] are proven for independent Verblunsky coefficients. Moreover, the results of [40] and [37] require rotational invariance of the distribution of the Verblunsky coefficients in the unit disk. By contrast, our results when mapped to the OPUC setting hold for strongly correlated random Verblunsky coefficients.

### 9.2. Open Problems

Instead of a conclusion section, in this final section we address a number of problems that still open in the theory of unitary Anderson models. These remarks are necessarily incomplete since we mainly concentrate on problems that stem from the current work.
The first problem is understanding the dynamical effects caused by the presence of the anomalous critical values proved in Theorem 3.2. In [25] it was shown that the presence of critical points with transfer matrices uniformly bounded in $n$ leads to super diffusive transport. However, as was shown in the proof of Theorem 3.2 the anomalies in the unitary Anderson model are characterized by transfer matrices $T_z(\omega, n)$ satisfying the asymptotics

$$\frac{1}{n} \mathbb{E} \left( \ln \| T_z(\omega, n) \| \right)^2 \rightarrow C > 0,$$

i.e., roughly, $\| T_z(\omega, z) \| \sim e^{(Cn)^{1/2}}$. To our knowledge the dynamical effects caused by such an anomaly have not been studied (they should be much weaker if detectable at all).

Another related question is localization of the model in case of i.i.d. phases possessing a singular distribution, in particular Bernoulli distribution. Spectral localization for the self-adjoint Anderson model with Bernoulli distribution was proven in [11] using positivity of the Lyapunov exponent and certain bounds on Green’s function. However, dynamical localization was proven under such assumption only as a consequence of variable-energy multi-scale analysis [42, 16]. As far as we know, no analogue of variable-energy multi-scale analysis has been developed for the unitary Anderson model and adopting such an important technique to the unitary setting seems to be an interesting problem.

Once pure point spectrum of the model is established, the next natural step is analyzing the local structure of the spectrum. In particular, what kind of correlation, if any, exists between eigenvalues? In the self-adjoint Anderson model with absolutely continuous distribution, it was proven that no such correlation exists and that the statistical distribution of eigenvalues converges locally to a stationary Poisson point process [31, 32]. A similar result was shown for the CMV matrices with uniformly distributed Verblunsky coefficients in [40]. In both cases fractional moment bounds of the type established in Theorem 7.1 played a major role. However, a rigorous study of eigenvalues statistics for the unitary Anderson model is still needed.

Localization of the unitary Anderson model in higher dimensions remains largely unsettled, even in the case of absolutely continuous distributions. So far, the only regime where localization has been established, in all dimensions, is that of high disorder, i.e.
small $t$, see [27] and Theorem 8.3. In the self-adjoint case localization is known to hold in a number of other regimes namely, weak disorder away from the spectrum of the unperturbed operator [1] and at band edges [19]. As far as we know none of these regimes has been studied in the unitary case and each remains an open problem requiring further study.
LIST OF REFERENCES


In this appendix we prove two Lemmas that have been used throughout the text. We start by using the properties of the Lyapunov exponent established in Chapter 3 to prove an estimate on the growth of solutions of \((U_\omega - z)\psi = 0\). The proof is almost identical to the one given for the self-adjoint case in Lemma 5.1 of [11], nevertheless we include it for completeness.

**Lemma A.1.** For a non-trivial measure \(\mu\) and for each compact subset \(\Lambda \in \mathbb{C}\) of quasi-energies with positive Lyapunov exponents, there exist \(\alpha = \alpha(\Lambda) > 0\) and \(0 < \delta = \delta(\Lambda) < 1\) and a positive integer \(C = C(\Lambda)\) such that

\[
\mathbb{E}[[|T_z(\omega, n)v|]|^{-\delta}] \leq Ce^{-\alpha n}
\]

for all \(z \in \Lambda\), \(n \geq 0\) and unit vector \(v \in \mathbb{C}^2\).

**Proof.** Recall that \(T_z(\omega, n) = T_z(\theta_{2(n-1)}^\omega, \theta_{2(n-1)+1}^\omega) \cdots T_z(\theta_0^\omega, \theta_1^\omega)\). In order to simplify the notation we denote \(T_{z,n} = T_z(\theta_{2(n-1)}^\omega, \theta_{2(n-1)+1}^\omega)\). Using that \(e^y \leq 1 + y + y^2e^{|y|}\), we obtain

\[
\mathbb{E}[[|T_z(\omega, n)v|]|^{-\delta}] \leq 1 - \delta\mathbb{E}[[\ln ||T_z(\omega, n)v||]]
\]

\[
+ \delta^2\mathbb{E}[[\ln ||T_z(\omega, n)v||]^2 \exp\{||\ln ||T_z(\omega, n)v||||\}].
\]

Using Hölder’s inequality, it follows that

\[
\mathbb{E}[[|T_z(\omega, n)v|]|^{-\delta}] \leq 1 - \delta\mathbb{E}[[|T_z(\omega, n)v||]]
\]

\[
+ \delta^2(\mathbb{E}[[|T_z(\omega, n)v||]^4])^{1/2}(\mathbb{E}[\exp\{\delta||\ln ||T_z(\omega, n)v||||\}]^2)^{1/2}.
\]
Since \(|\det T_{z,k}| = 1\), then \(||T_{z,k}|| = ||T_{z,k}^{-1}||\) and
\[
\ln ||T_z(\omega, n)v|| \leq \sum_{k=1}^{n} \ln ||T_{z,k}||.
\]
Since the phases \(\theta_k^\omega\) are i.i.d., then
\[
\mathbb{E}[||T_z(\omega, n)v||^{-\delta} ] \leq 1 - \delta \mathbb{E}[\ln ||T_z(\omega, n)v||] + \delta^2 n(\mathbb{E}[\ln ||T_{z,k}||])^{1/2}(\mathbb{E}[||T_{z,k}||^{2\delta}])^{n/2}.
\]
Since \(\Lambda\) is a compact subset, then there exist \(C_1, C_2\) such that For all \(z \in \Lambda\) and all \(0 < \delta \leq 1\)
\[
\mathbb{E}[||T_z(\omega, n)v||^{-\delta} ] \leq 1 - \delta \mathbb{E}[\ln ||T_z(\omega, n)v||] + \delta^2 C_1 C_2^n.
\]
By Theorem 3.3, \(\gamma(z)\) is continuous on \(\Lambda\), thus \(\gamma = \inf \{\gamma(z) : z \in \Lambda\} > 0\) and
\[
\gamma_\omega(z) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\ln ||T_z(\omega, n)v||]
\]
converges uniformly in \(z \in \Lambda\) and all unit vectors \(v \in \mathbb{C}^d\). Thus we can choose \(n_0 = n_0(\Lambda)\) and \(\delta\) small enough such that for some \(\varepsilon > 0\)

(A.1) \[
\mathbb{E}[||T_z(\omega, n_0)v||^{-\delta} ] \leq 1 - \varepsilon.
\]
For any \(n \in \mathbb{N}\) we have that \(n = kn_0 + r\) for some \(k \in \mathbb{N}_0\) and \(0 \leq r < n_0\). Therefore,
\[
\mathbb{E}[||T_z(\omega, n)v||^{-\delta} ] \leq \mathbb{E}[||T_z, kn_0+1^{-1} \ldots [T_z, n]^{-1}||^{\delta} \mathbb{E}[||T_z, kn_0 \ldots T_z, 1v||^{-\delta}]
\]
\[
\leq \{\mathbb{E}[||T_z, 1||^{-1}]\}^{n_0}
\]
\[
\times \mathbb{E}[||T_z, kn_0 \ldots T_z, (k-1)n_0+1^{-1} T_z, (k-1)n_0 \ldots T_z, 1v||^{-\delta} \mathbb{E}[||T_z, (k-1)n_0 \ldots T_z, 1v||^{-\delta}].
\]
Since the bound in (A.1) is independent of the unit vector \(v\), it follows that
\[
\mathbb{E}[||T_z(\omega, n_0)v||^{-\delta} ] \leq \{\mathbb{E}[||T_z, 1||^{-1}]\}^{n_0}(1 - \varepsilon) \mathbb{E}[||T_z, (k-1)n_0 \ldots T_z, 1v||^{-\delta}]
\]
\[
\leq C(1 - \varepsilon) \mathbb{E}[||T_z, (k-1)n_0 \ldots T_z, 1v||^{-\delta},
\]
where \(C = \sup \{\mathbb{E}[||T_z, 1||^{-1}]\}^{n_0} < \infty\). Repeating this argument we obtain
\[
\mathbb{E}[||T_z(\omega, n)v||^{-\delta} ] \leq C(1 - \varepsilon)^k \leq e^{-\alpha_1 n},
\]
for some $\alpha_1 = \alpha_1(\Lambda)$, $n_1 = n_1(\Lambda)$, all $n \geq n_1$ and all $z \in \Lambda$. Now, for $n \leq n_1$, using again that the phases are i.i.d. along with the fact that $||T_z^{-1}|| \geq 1$, it follows that

$$\mathbb{E}[||T_z(\omega, n) v||^{-\delta}] \leq \{\mathbb{E}(||T_z^{-1}||)^n\}^{n_1}.$$  

Since $\Lambda$ is a compact subset, taking $C_1(\Lambda) = \sup_{z \in \Lambda} \{\mathbb{E}(||T_z^{-1}||)^{n_1} e^{\alpha_1 n}\} < \infty$ allows us to conclude that

$$\mathbb{E}[||T_z(\omega, n) v||^{-\delta}] \leq C_1(\Lambda) e^{-\alpha_1 n}.$$  

Taking $C = \max\{1, C_1\}$ gives the required result. \(\square\)

The next result was proven in [27], nevertheless we include the proof here for the convenience of the reader.

**Lemma A.2.** Assume $\mu$ is a probability measure on $\mathbb{T}$ such that $d\mu(\theta) = \tau(\theta)d\theta$, where $\tau \in L^\infty(\mathbb{T})$. Then for any $0 < s < 1$, there exists $0 < C_\mu(s) < \infty$ such that for all $\beta \in \mathbb{C}$

$$\int_T d\mu(\theta) \frac{1}{|e^{i\theta} - \beta|^s} \leq C_\mu(s).$$

**Proof.** For any $\lambda > 0$ we have that

$$\int_T d\mu(\theta) \frac{1}{|e^{i\theta} - \beta|^s} \leq \lambda \int_{|e^{i\theta} - \beta|^{-s} \leq \lambda} d\mu(\theta) + ||\tau||_\infty \int_{|e^{i\theta} - \beta|^{-s} \geq \lambda} \frac{1}{|e^{i\theta} - \beta|^{-s}} d\theta$$

(A.2)

$$\leq \lambda + ||\tau||_\infty \int_\lambda^\infty \left| \{ |e^{i\theta} - \beta|^{-s} \geq x \} \right| dx,$$

where $|A|$ denote the Lebesgue measure of the set $A$. Since

$$\left| \{ |e^{i\theta} - \beta|^{-s} \geq x \} \right| = \int_{|e^{i\theta} - \beta| \leq x^{-1/s}} d\theta,$$

the latter integral becomes smaller if $\beta$ is replaced by its real part, so we can assume without loss of generality that $\beta \in \mathbb{R}$. Evaluating the integral gives the arclength of the intersection of the unit circle with a circle of radius $1/x^{1/s}$ centered at $\beta$. For $x^{-1/s} \geq 1,$
the integral takes its maximum value $2\pi$, when $\beta = 0$. On the other hand, when $x^{-1/s} \leq 1$, the integral is maximized by $\beta = \sqrt{1 - x^{-2/s}}$ and takes the value
\[
\int_{\{e^{i\theta} - \beta \leq 1/x^{1/s}\}} d\theta = 2 \arcsin \left(1/x^{1/s}\right).
\]
Therefore, (A.2) now gives
\[
\int_{\mathbb{T}} d\mu(\theta) \frac{1}{|e^{i\theta} - \beta|^s} \leq \lambda + 2\pi ||\tau||_{\infty} + 2||\tau||_{\infty} \int_{\lambda}^{\infty} \arcsin \left(1/x^{1/s}\right) dx.
\]
Minimizing this upper bound on $\lambda$ is achieved by choosing $\lambda$ such that
\[
1 - 2||\tau||_{\infty} \arcsin \left(1/\lambda^{1/s}\right) = 0.
\]
Noting that $||\tau||_{\infty} \geq 2\pi$, due to normalization of $\mu$, the minimizer is given by
\[
\lambda_0 = \left[\sin \frac{1}{2||\tau||_{\infty}}\right]^{-s} > 1.
\]
It follows that
\[
\int_{\mathbb{T}} d\mu(\theta) \frac{1}{|e^{i\theta} - \beta|^s} \leq \lambda_0 + 2\pi ||\tau||_{\infty} + 2||\tau||_{\infty} \int_{\lambda_0}^{\infty} \arcsin \left(1/x^{1/s}\right) dx
\]
\[
= C_\mu(s) < \infty,
\]
where the last inequality follows from the fact that as $x \to \infty$ the latter integrand behaves like $1/x^{1/s}$ which is integrable for $0 < s < 1$. This gives the required estimate for $e^{i\theta}$, the same estimate holds for $e^{-i\theta}$ by conjugation. \qed
A UNITARY DIMER MODEL

In this section we study a unitary version of the Dimer model, which is obtained from the Anderson model (2.5) by doubling up the random phases. More rigorously, let a probability measure $\mu$ on $\mathbb{T}$ be given, define $g : \mathbb{T} \to \mathbb{T}^2$ as $g(\theta) = (\theta, \theta)$ and let $\tilde{\mu}$ be the probability measure supported on the diagonal of $\mathbb{T}^2$ induced by $\mu$ through $g$: $\tilde{\mu}(\tilde{B}) = \mu(g^{-1}(\tilde{B}))$ for Borel sets $\tilde{B}$ in $\mathbb{T}^2$. We introduce the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, where $\tilde{\Omega}$ is identified with $(\mathbb{T}^2)^\mathbb{Z}$, $\tilde{\mathcal{F}}$ is the $\sigma$-algebra generated by cylinders of Borel sets in $\mathbb{T}^2$ and $\tilde{\mathbb{P}} = \bigotimes_{k \in \mathbb{Z}} \tilde{\mu}$.

For $\omega = (\omega_k)_{k \in \mathbb{Z}} \in \tilde{\Omega}$, the random phases $\theta_n^\omega, n \in \mathbb{Z}$, used in (2.3) and (2.5) to define $U_\omega$ are now chosen as

$$(\theta_{2k}, \theta_{2k+1}) = \omega_k, \quad k \in \mathbb{Z}.$$ 

For this model we again prove that the almost sure absolutely continuous spectrum is empty. We also show that the case of a Bernoulli measure

$$\mu = p\delta_a + q\delta_b, \quad p + q = 1, \quad a, b \in \mathbb{T}, \quad a \neq b,$$

(B.1) gives rise to additional critical quasi-energies, as this is the case of least randomness.

The following theorem states that for any non-trivial distribution $\mu$ on $\mathbb{T}$, i.e. $\{a, b\} \subset \text{supp } \mu$ for $a \neq b$, the Lyapunov exponent is positive for all but a finite set of quasi-energies, given by

$$M := \{-a, -b\} \cup (M_a \cap M_b),$$

where

$$M_a = \{\arccos(r^2) - a, 2\pi - \arccos(r^2) - a, \arccos(r^2 - t^2) - a, 2\pi - \arccos(r^2 - t^2) - a\},$$

$$M_b = \{\arccos(r^2) - b, 2\pi - \arccos(r^2) - b, \arccos(r^2 - t^2) - b, 2\pi - \arccos(r^2 - t^2) - b\}.$$
An immediate consequence is that the almost sure absolutely continuous spectrum of these operators is trivial.

**Theorem B.1.** If \( \{a, b\} \subset \text{supp} \mu \), then for all \( \lambda \in \mathbb{T} \setminus M \), the Lyapunov exponent \( \gamma(e^{i\lambda}) \) is strictly positive. In particular, \( \Sigma_{ac} = \emptyset \).

As before, we will use Fürstenberg’s Theorem [7] to prove positivity of Lyapunov exponents. Let \( \tilde{G}_{\lambda, \mu} \) be the closed group corresponding to \( G_{e^{i\lambda}, \mu} \) from the previous section.

We will show that \( \tilde{G}_{\lambda, \mu} \) is both non-compact and strongly irreducible for all \( \lambda \) outside of \( M \). As both of these properties carry over to larger groups, we may assume for the rest of the proof of Theorem B.1 that \( \text{supp} \mu = \{a, b\} \). Thus \( \tilde{G}_{\lambda, \mu} \) is generated by just two matrices, \( T(\theta, \theta) \) and \( T(\eta, \eta) \), where \( \theta = a + \lambda, \eta = b + \lambda \).

In order to prove Theorem B.1, we start by mapping the problem into a somewhat simpler form. In order to simplify the notation, we again let \( x := e^{-i\theta} \), \( z := e^{-i\eta} \), and let \( \rho, \rho_1 \) be the two eigenvalues of \( T(\theta, \theta) \). Since \( \text{tr} T(\theta, \theta) = \frac{2r^2}{t^2} - \frac{1}{t^2} (x + \bar{x}) \), and \( \det T(\theta, \theta) = \det T(\eta, \eta) = 1 \), we have the following cases

\[
\begin{align*}
\rho &= \frac{1}{\rho_1} > 1, \quad \text{when } |\text{tr} T(\theta, \theta)| > 2, \\
\rho &= \rho_1, \rho^2 = 1, \quad \text{when } |\text{tr} T(\theta, \theta)| = 2.
\end{align*}
\]

This allows us to introduce the transformation \( N \) given by

\[
N = \begin{pmatrix}
\frac{r}{t}(1 - x) & x + \rho \\
x + \rho & -\frac{r}{t}(1 - x)
\end{pmatrix}.
\]

Using that \( (x + \rho)(x + \rho_1) = -\frac{r^2}{t^2}(1 - x)^2 \), we deduce that \( \det N = (x + \rho)(\rho_1 - \rho) \). Therefore, \( N \) is invertible as long as \( |\text{tr} T(\theta, \theta)| \neq 2 \). Moreover,

\[
E = NT(\theta, \theta)N^{-1} = \begin{pmatrix}
\rho & 0 \\
0 & 1/\rho
\end{pmatrix}.
\]
A short calculation shows that the elements of \( F = NT(\eta, \eta)N^{-1} \) are given by

\[
F_{11} = \frac{1}{t^2(\rho - 1/\rho)} [2r^2(1 + \rho) - (z\bar{x} + \bar{z}x) - \rho(z + \bar{z})], \\
F_{12} = F_{21} = \frac{2ir}{t^2(\rho - 1/\rho)} [\Im(x) - \Im(z) + \Im(z\bar{x})], \\
F_{22} = -\frac{1}{t^2(\rho - 1/\rho)} [2r^2(1 + 1/\rho) - (z\bar{x} + \bar{z}x) - 1/\rho(z + \bar{z})].
\]

(B.4)

Notice that since \( \theta \neq \eta \), \( F_{12} = 0 \) if and only if either \( \eta = 0 \) or \( \theta = 0 \).

Since proving non-compactness and strong irreducibility of \( \tilde{G}_{\lambda,\mu} \) is equivalent to proving the same properties for the group \( \tilde{H}_{\lambda,\mu} \) generated by the matrices \( E, F \), we will use the latter, somewhat simpler matrices whenever it helps simplifying the proofs.

**Lemma B.1.** For all \( \lambda \in \mathbb{T}\setminus\{-a, -b\} \), the group \( \tilde{G}_{\lambda,\mu} \) is non-compact.

**Proof.** Since \( \lambda \neq -a \), we have that \( \theta \neq 0 \) and thus \( trT(\theta, \theta) \neq -2 \). Therefore, the preceding discussion suggests the proof should be divided into the following cases:

**Case I:** \( tr T(\theta, \theta) = 2 \). By (B.2) we have \( \rho = \rho_1 = 1 \) and since by definition \( T(\theta, \theta) \neq I \) it follows that there exists a non-singular matrix \( R \) such that

\[
RT(\theta, \theta)R^{-1} = \begin{pmatrix} \rho & 1 \\ 0 & \rho \end{pmatrix}.
\]

Since \( \|RT(\theta, \theta)R^{-1}\|^n \) grows with \( n \), the group generated by \( RT(\theta, \theta)R^{-1}, RT(\eta, \eta)R^{-1} \) is non-compact, which implies that \( G_{\lambda,\mu} \) is non-compact.

**Case II:** \( |tr T(\theta, \theta)| > 2 \), again by (B.2), \( T(\theta, \theta) \) has an eigenvalue \( \rho > 1 \) which gives the required result.

**Case III:** \( |tr T(\theta, \theta)| < 2 \). In this case equations (B.2), (B.3) give

\[
E = \begin{pmatrix} e^{iy} & 0 \\ 0 & e^{-iy} \end{pmatrix},
\]

with \( y = \arccos\left(\frac{r^2}{t^2} - \frac{1}{2t^2}(x + \bar{x})\right) \), and \( y \in (0, \pi) \). Equations (B.4) lead to

\[
F = \begin{pmatrix} \alpha e^{ic} & \beta \\ \beta & \alpha e^{-ic} \end{pmatrix}, \quad \alpha \geq 0, \beta \in \mathbb{R}, c \in \mathbb{T}.
\]
Using that $\det F = 1$, it follows that $\alpha > 0$.

Now we follow a strategy outlined in [15] to show that there exists a sequence of elements in $\tilde{H}_{\lambda,\mu}$ with unbounded norms. In order to do so, we note that any element of $P(\mathbb{C}^2)$ can be written in the form

$$e_{(u,v)} = \begin{pmatrix} e^{iu}\cos(v) \\ e^{-iu}\sin(v) \end{pmatrix}, \quad (u,v) \in [0,\pi) \times [0,\pi).$$

Therefore, for any element $e_{(u,v)}$ of $P(\mathbb{C}^2)$ we have

$$Fe_{(u,v)} = \begin{pmatrix} \alpha e^{i(c+u)}\cos(v) + \beta e^{-iu}\sin(v) \\ \alpha e^{-i(c+u)}\sin(v) + \beta e^{iu}\cos(v) \end{pmatrix}. $$

Using that $\alpha^2 - \beta^2 = 1$, we get

$$||Fe_{(u,v)}||^2 - 1 = 2\beta^2 + 4\alpha\beta \cos(2u+c) \cos(v) \sin(v).$$

Since $\beta = \frac{r}{t^3} \Im(x) - \Im(z) + \Im(zx)$ with distinct $x,z$ and neither equals 1 under the current assumptions, we see that $\beta \neq 0$.

In the case $\beta > 0$: If $\cos(v) \sin(v) = 0$, then $||Fe_{(u,v)}||^2 - 1 > \beta^2$ for all $u \in [0,\pi)$. While for $\cos(v) \sin(v) > 0$, $||Fe_{(u,v)}||^2 - 1 > \beta^2$ is equivalent to

$$\cos(2u+c) > \frac{-\beta}{4\alpha \cos(v) \sin(v)}.$$ 

In particular, the condition $\cos(2u+c) > \frac{-\beta}{4\alpha}$ guarantees that $||Fe_{(u,v)}||^2 > 1 + \beta^2$. Defining $K_+ := \{u \in [0,\pi) : \cos(2u+c) > \frac{-\beta}{4\alpha}\}$, we see that $|K_+| > \pi/2$ and for all $u \in K_+$ we have $||Fe_{(u,v)}||^2 > 1 + \beta^2$.

Similarly, for $\cos(v) \sin(v) < 0$, let $K_- := \{u \in [0,\pi) : \cos(2u+c) < \frac{\beta}{4\alpha}\}$. Then for all $u \in K_-$, we have $||Fe_{(u,v)}||^2 > 1 + \beta^2$ and $|K_-| > \pi/2$.

Hence, given any $v \in [0,\pi)$, there exists an interval $K_v \subset [0,\pi)$, i.e. an interval in $\mathbb{R}\setminus\pi\mathbb{Z}$, such that $|K_v| > \pi/2$ and $||Fe_{(u,v)}||^2 > 1 + \beta^2$, for all $u \in K_v$. Therefore, starting with an appropriately chosen vector $e_{(u,v)} \in P(\mathbb{C}^2)$ such that $||Fe_{(u,v)}||^2 > 1 + \beta^2$, applying $F$ will result in a vector $ce_{(u_1,v_1)}$ with $c > 1$. Now we apply $E$ as many times as required to get a vector $ce_{(\tilde{u},v_1)}$ (or $-ce_{(\tilde{u},v_1)}$) such that $\tilde{u} \in K_{v_1}$. Iterating this process gives a
sequence of vectors with unbounded norms. The case $\beta < 0$ is treated similarly. Thus we have proved the non-compactness of $\tilde{H}_{\lambda,\mu}$, and consequently that of $\tilde{G}_{\lambda,\mu}$. □

The next step is proving that $\tilde{G}_{\lambda,\mu}$ is strongly irreducible for all $\lambda$ outside the set $M$.

**Lemma B.2.** $\tilde{G}_{\lambda,\mu}$ is strongly irreducible, for all $\lambda \in \mathbb{T} \setminus M$.

**Proof.** Since we already proved that $\tilde{G}_{\lambda,\mu}$ is non-compact (Lemma B.1), it suffices to show that for all $v \in \mathcal{P}(\mathbb{C}^2)$, $\# \{ gv : g \in \tilde{G}_{\lambda,\mu} \} \geq 3$.

We first note that $\rho^4 = 1$ implies that $|\text{tr}T(\theta, \theta)| \in \{0, 2\}$, which in turn implies that $\lambda \in \{-a, -b\} \cup M_a$. Therefore, the condition $\lambda \notin \{-a, -b\} \cup M_a$ gives that $\rho^4 \neq 1$. Hence, $\{I, E, E^2\} \subset \tilde{G}_{\lambda,\mu}$ maps every $v \in \mathcal{P}(\mathbb{C}^2)$ to three different directions unless $v$ coincides with either $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. At this point, we note that the diagonal elements of the matrix $F$, given in (B.4), vanish simultaneously only if $\lambda \in \{-a, -b\}$. Thus, iterations of the operator $F$, followed if necessary with iterations of $E$, maps each of the latter directions to at least three different elements of $\mathcal{P}(\mathbb{C}^2)$. This proves strong irreducibility of the group $\tilde{H}_{\lambda,\mu}$, for all $\lambda \notin \{-a, -b\} \cup M_a$, which gives the corresponding result for $\tilde{G}_{\lambda,\mu}$. A similar argument, replacing the rules of $a$ and $b$, gives a similar assertion for $\lambda \notin \{-a, -b\} \cup M_b$, thus finishing the proof. □

**Proof of Theorem B.1.** The previous two lemmas combined with Fürstenberg’s Theorem, immediately give $\gamma(\lambda) > 0$ for all $\lambda \in \mathbb{T} \setminus M$. Using Theorem 2.3, we deduce that $\Sigma_{ac} = \emptyset$. □

Even though the finiteness of the set $M$ is more than enough to prove the absence of absolutely continuous spectrum for the unitary dimer model, $M$ is by no means optimal. Determining whether or not a certain element of $M$ is, in fact, a critical quasi-energy of $U_\omega$ requires further analysis. Nevertheless, the proof suggests that if the support of $\mu$ contains three or more points than generically $M = \emptyset$. Even for the Bernoulli unitary dimer model where $\text{supp}\mu = \{a, b\}$ the analysis is likely to fall into a number of different sub-cases. However, for generic choices of $a, b$ we see that $M_a \cap M_b = \emptyset$, thus $M = \{-a, -b\}$ and
indeed the situation where $\lambda \in \{-a, -b\}$ is readily accessible. Guided by the proof of Theorem 2.2(i) of [17] for the self-adjoint dimer model we prove that,

**Proposition B.1.** For a probability measure $\mu$ given by (B.1), we have the following:

(i) If $|a-b| \in \sigma(S)$, then $\gamma(-a) = \gamma(-b) = 0$.

(ii) If $|a-b| \in \rho(S)$, then both $\gamma(-a) > 0$ and $\gamma(-b) > 0$.

**Proof.** For $\lambda = -a$, we have that $T(\theta, \theta) = -I$, and $\eta = b - a$. Since $\text{tr} T(\eta, \eta) = \frac{2r^2}{t^2} - \frac{2\cos(\eta)}{t^2}$, then $|\text{tr} T(\eta, \eta)| \leq 2$ when $b - a \in \sigma(S)$ and $|\text{tr} T(\eta, \eta)| > 2$ if $b - a \in \rho(S)$.

Now, let $m_n := \#\{k : 1 \leq k \leq n, \theta_{2k} = b\}$, then $\mathbb{P}$-almost surely

$$\lim_{n \to \infty} \frac{m_n}{n} = q.$$ 

This along with the fact that

$$\lim_{m_n \to \infty} ||[T(\eta, \eta)]^{m_n}||^{1/m_n} = \max_{1 \leq i \leq 2} |r_i|,$$

where $r_i$ are the eigenvalues of $T(\eta, \eta)$, gives the results of the proposition. \qed